Solving for the Retirement Age in a Continuous-time Model with Endogenous Labor Supply

Emin Gahramanov* Xueli Tang**

*Deakin University
**Deakin University

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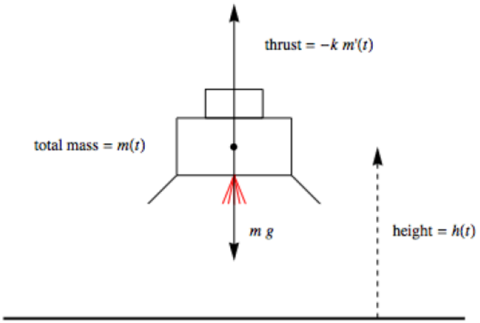
  - the module is moving with some downward velocity. It falls under a uniform gravitational acceleration and there is no atmosphere to resists its motion.
  - Velocity can be controlled by engines that can exert some maximum thrust in either the upward or downward position.
  - The cost function is the total fuel consumed, which is proportional to the time integral of the magnitude of the thrust exerted by the engines.

- Mureșan (2014): assume the spacecraft is near the Moon.



- t is time
- $m(t)$ is the mass of the spacecraft, which varies as fuel is burned
- $m'(t)$ is the rate of change of mass, constrained by $-\mu \le m'(t) \le 0$
- $g = 1.63$, the gravitational constant near the Moon
- k is a constant, the relative velocity of the exhaust gases with respect to the spacecraft
- $T(t) = -k\,m'(t)$, the thrust
- $h(t)$ is the the height, with $h(t) \ge 0$
- $v(t) = h'(t)$, the velocity
- $u(t) = m'(t)$, the control function

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- The *Maximum Principle* for optimal control, developed in the late 1950s by **L.S. Pontryagin** and his co-workers, is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls.

The simplest control problem in its general form is to

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt \tag{1}$$

subject to

$$\frac{dx(t)}{dt} = g(t, x(t), u(t)), \tag{2}$$

$$t_0, t_1, x(t_0) = x_0 \text{ fixed; } x(t_1) \text{ free.} \tag{3}$$

Here f and g are assumed to be known and continuously differentiable functions of three independent arguments, none of which is a derivative. The control variable $u(t)$ must be a piecewise continuous function of time. The state variable is $x(t)$. The control u influences the objective (1) both directly and indirectly through its impact on the evolution of the state variable.

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  - The solutions presented in the above studies were *not* complete.

- Consider an intertemporal utility maximization problem by a representative agent. Preferences are

$$U(c(t), l(t)) = \frac{(c(t)^{\phi} l(t)^{1-\phi})^{1-\sigma}}{1-\sigma}. \tag{4}$$

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- The agent's problem can be formulated as

$$\underset{\{c(t), l(t)\}}{Max} \int_0^T Q(t) e^{-\rho t} U(c(t), l(t)) dt \tag{5}$$

subject to:

$$\frac{dk(t)}{dt} = rk(t) + w\epsilon(t)(1 - l(t)) - c(t), \quad \text{for } t \in [0, T], \tag{6}$$

$$0 \leq l(t) \leq 1, \tag{7}$$

$$k(0) = 0, \tag{8}$$

$$k(T) = 0. \tag{9}$$

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- The problem is quite difficult and lengthy to solve.
  - We find: for a disturbingly large number of parameters so commonly used in related studies, the above problem does *not* have a solution so commonly assumed.

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  - **Step 1**. Make a *conjecture* about the structure of the solution in case if it is not clear what the structure should be.
  - **Step 2**. Seek a path with this structure satisfying the necessary conditions.
- If Step 2 fails, one needs to investigate *alternative* solution structures.

- Back to our problem...

Let us define the Hamiltonian function

$$\begin{aligned} H_1 &= Q(t)e^{-\rho t}\frac{(c(t)^{\phi}l(t)^{1-\phi})^{1-\sigma}}{1-\sigma} \\ &\quad +\mu(t)(rk(t)+w\epsilon(t)(1-l(t))-c(t)). \end{aligned} \tag{10}$$

Optimal controls must be chosen so the following conditions are satisfied:

$$\frac{dk(t)}{dt} = \frac{\partial H_1}{\partial \mu(t)} = rk(t) + w\epsilon(t)(1-l(t)) - c(t), \tag{11}$$

$$\frac{d\mu(t)}{dt} = -\frac{\partial H_1}{\partial k(t)} = -\mu(t)r, \tag{12}$$

and

$$\max_{\{1-l(t)\geq 0, c(t)\}} H_1. \tag{13}$$

Necessary condition is that there exists a time-dependent multiplier $\lambda(t) \leq 0$, so that if **the Lagrangian of the Hamiltonian**

$$H = H_1 + \lambda(t)(l(t) - 1), \tag{14}$$

then the Lagrangian of the Hamiltonian is stationary with respect to the controls, and *the complimentary slackness condition* holds, implying

$$\frac{\partial H}{\partial c(t)} = 0, \tag{15}$$

$$\frac{\partial H}{\partial l(t)} = 0, \tag{16}$$

$$\lambda(t)(l(t) - 1) = 0, \tag{17}$$

$$1 - l(t) \geq 0. \tag{18}$$

Let us guess that $l(t) < 1$ for $t \in [0, t^*)$, and $l(t) = 1$ for $t \in [t^*, T]$. Complementarity condition implies that if $\lambda(t) = 0$, then $l(t) < 1$, and we have the system of equations

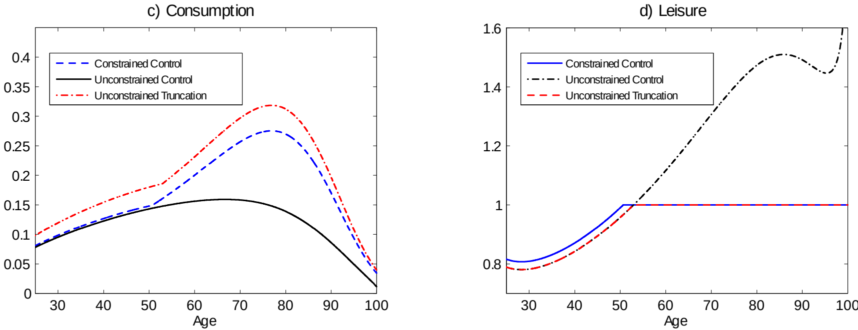
$$\begin{pmatrix} k(t) \\ \mu(t) \end{pmatrix}' = \begin{pmatrix} rk(t) + w\epsilon(t)(1 - l(t)) - c(t) \\ -\mu(t)r \end{pmatrix} \tag{19}$$

for $t \in [0, t^*)$. Similarly, if $\lambda(t) < 0$, then $l(t) = 1$, and we have the system of differential equations

$$\begin{pmatrix} k(t) \\ \mu(t) \end{pmatrix}' = \begin{pmatrix} rk(t) - c(t) \\ -\mu(t)r \end{pmatrix} \text{ for } t \in [t^*, T]. \tag{20}$$

Hence, the solution to the problem can be found by piecing together the solution of (19)-(20). The switch point t^* is deduced from the state continuity condition.

- A Numerical Example:



$$\sigma = 3,\ \rho = 1\%,\ \phi = 0.1,\ r = 5.5\%$$

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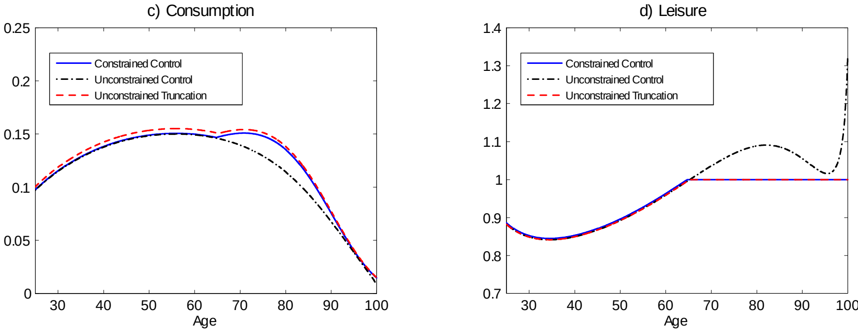

$$\sigma = 3.7,\ \rho = 2\%,\ r = 4.5\%,\ \phi = 0.11$$

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    - **Gruber and Wise (2005)**: raising the age of retirement eligibility in the United States by one year would lead to about a 10 percent savings to the system.
- The existence of the solution we are interested in is highly sensitive to the gap between the time discount and interest rates. Even if such gap is small, by varying other parameters of the model, we may run to this problem again.

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- *Pseudospectral optimal control* is a joint theoretical-computational method for solving optimal control problems. It combines **pseudospectral (PS) theory** with optimal control theory.
    - PS methods are a class of numerical methods used in applied mathematics and scientific computing for the solution of partial differential equations. They are closely related to spectral methods, but complement the basis by an additional pseudo-spectral basis, which allows to represent functions on a quadrature grid.

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- The Radau pseudospectral method is an orthogonal collocation Gaussian quadrature implicit integration method where collocation is performed at the Legendre-Gauss-Radau points.

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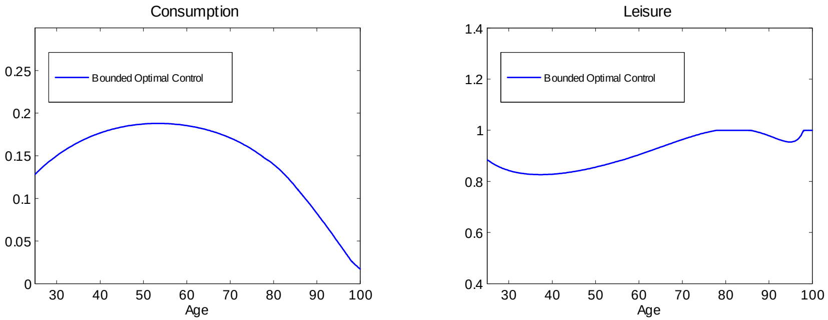

$$\sigma = 4,\ \rho = 3\%,\ \phi = 0.14,\ r = 5\%$$

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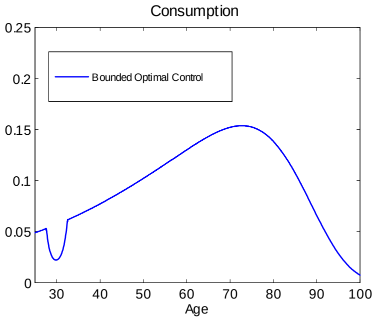

$$\sigma = 0.01,\ \rho = 1\%,\ \phi = 0.05,\ r = 4\%$$

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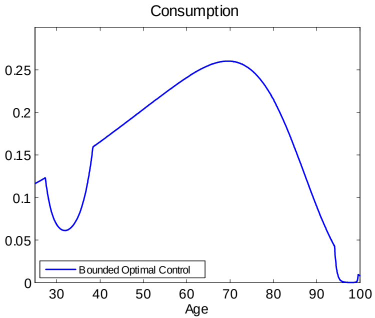

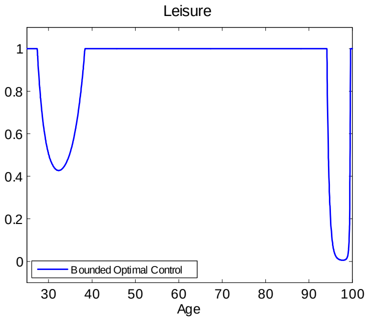

$$\sigma = 0.04,\ \rho = 1.3\%,\ \phi = 0.11,\ r = 3.5\%$$

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  - It is very possible to have "unconventional" sequences of optimal arcs and very "lumpy"/"awkward" consumption profiles, with multiple entries and exists to the job market even in a simple model with standard parameters and no frictions.

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- Some further possible questions:
  - How to make sure people do not jump in and jump out the job market as we saw? Maybe consider the costs of getting back to work after a long time of being unemployed?
  - Do people have to experience frictions or some unexpected events to tolerate career interruptions? They can interrupt their careers simply because they love taking a break once in a while!

THANK YOU!