A Note on Isotone Solutions of the Parametric Linear Complementarity Problem

by

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A Note on Isotone Solutions of the Parametric Linear Complementarity Problem

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For a given matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem $LCP(q, M)$ is that of finding $w, z \in \mathbb{R}^n$ such that

**Abstract.** This paper shows that the parametric linear complementarity problem $w = Mz + q + \alpha p$, $w \geq 0$, $z \geq 0$, $w^Tz = 0$, $\alpha \geq 0$ has isotone complementary solutions for $q = 0$ and every $p$ iff $M$ is a $P$-matrix. Thus, isotonicity for every $q \geq 0$ and every $p$ reduces to monotonicity where $M$ is a $P$-matrix. By excluding $q = 0$, it is shown that isotonicity is possible for every $0 \neq q \geq 0$ and every $p$ where $M$ is not a $P$-matrix.

**Keywords.** Parametric linear complementarity problem; isotone complementary solutions; matrices

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1. Introduction

For a given matrix $M \in \mathbb{R}^{m \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem $LCP(q, M)$ is that of finding $w, z \in \mathbb{R}^n$ such that

$$w = Mz + q, \quad w \geq 0, \quad z \geq 0, \quad w^T z = 0. \quad (1)$$

A pair $(w; z)$ that satisfies (1) is called a complementary solution and the set of complementary solutions of the $LCP(q, M)$ is denoted by $C(q, M)$. The set of all $q \in \mathbb{R}^n$ for which the $LCP(q, M)$ has a complementary solution is denoted by $K(M)$.

For a given matrix $M \in \mathbb{R}^{m \times n}$ and vectors $q, p \in \mathbb{R}^n$, the parametric linear complementarity problem $PLCP(q+\alpha p, M)$ consists of the family of linear complementarity problems \( \{LCP(q+\alpha p, M) : \alpha \geq 0\} \), where the parameter $\alpha \in \mathbb{R}$. The PLCP arose in the analysis of elastoplastic structures (Maier [7]) and has also found applications in other areas such as the computation of economic equilibria (Benveniste [1]; Pang and Lee [11]), portfolio selection (Pang [9]), and actuarial graduations (Pang, Kaneko, and Hallman [10]).
As proposed by Maier [7], the PLCP(q+αp,M) assumes q > 0 and is concerned with determining conditions under which the z-component of the complementary solution \((w(α);z(α))\) of the LCP(q+αp,M) has coordinates that are monotone nondecreasing with respect to α. When every coordinate of \(z(α)\) is monotone nondecreasing, the vector function \(z(α)\) is also said to be monotone nondecreasing.

When \(M\) is a P-matrix (i.e., \(M\) has positive principal minors), the LCP(q,M) has a unique complementary solution for each \(q \in \mathbb{R}^n\) (Murty [8]). Thus, the monotonicity of \(z(α) = z(α;q,p)\) is well-defined since \(z(α)\) is a point-to-point mapping. Under the assumption that \(M\) is a P-matrix and \(q \geq 0\), Cottle [2] proved the following theorem:

**Theorem 1.1.** (Cottle [2]) Given the PLCP(q+αp,M) where \(M\) is a P-matrix. Then \(z(α) = z(α;q,p)\) is monotone nondecreasing for every \(q \geq 0\) and every \(p \geq 0\) iff \(M\) is a Minkowski matrix (i.e., a P-matrix with nonpositive off-diagonal entries).

In view of the importance of the uniqueness of complementary solutions for the monotonicity of \(z(α)\) to be well-defined the question arises: Are there matrices \(M\) other than the P-matrices for which the LCP(q,M) has a unique complementary solution for every
q ∈ K(M)? The answer is that there are none. For if the LCP(q, M) has a unique complementary solution for every q ∈ K(M), then it has a unique complementary solution for every q ≥ 0 since the nonnegative orthant of R^n is always a subset of K(M). Consequently, M is an Lₜ-matrix (Eaves [5]) which implies that M is a Q-matrix or, equivalently, K(M) = R^n (Eaves [5]; Cottle and Dantzig [3]); hence, M is a P-matrix.

When M is not a P-matrix, the LCP(q+αp, M) may not have a complementary solution and when it has, the complementary solution may not be unique. Thus z(α) becomes a point-to-set mapping. In this case, Kaneko [6] proposed a more general definition of monotonicity. Let

\[ T = \{α ≥ 0 | C(q+αp, M) ≠ ∅}\]

and define the functions

\[ z: T → R^n, \]

where z(α) is the z-component of an element of C(q+αp, M). Following Kaneko [6] we refer to these functions as complementary maps and adopt his generalized definition of monotonicity.

**Definition 1.1.** The PLCP(q+αp, M) is said to have an isotone complementary map iff there exists a complementary map z(α) such that z_j(α) is monotone nondecreasing with respect to α ∈ T for each j = 1, 2, ..., n.
Definition 1.2. The PLCP\((q+αp,M)\) is said to have isotone complementary solutions iff every complementary map \(z(α)\) is isotone with respect to \(α ∈ T\).

Remark 1.1. When \(M\) is a P-matrix, isotonicity coincides with monotonicity since there is only one complementary map.

Under the assumption that \(M\) is a Z-matrix (i.e., \(M\) has nonpositive off-diagonal entries), Kaneko [6] proved the following theorem:

Theorem 1.2 (Kaneko [6]) Let \(M\) be a Z-matrix. The PLCP\((q+αp,M)\) has isotone complementary solutions for every \(q ≥ 0\) and every \(p\) iff \(M\) is a Minkowski matrix.

From Theorems 1.1 and 1.2, we see that isotonicity for every \(q ≥ 0\) and every \(p\) when \(M\) is a Z-matrix reduces to monotonicity for every \(q ≥ 0\) and every \(p\) with \(M\) being a P-matrix. This paper drops the assumption that \(M\) is a Z-matrix and proves that, at \(q = 0\), the PLCP\((0+αp,M)\) has isotone complementary solutions for every \(p\) iff \(M\) is a P-matrix. It follows that a necessary condition for isotonicity for every \(q ≥ 0\) and every \(p\) is that \(M\) be a P-matrix. (This is the necessary condition in Theorem 1.2 which was proved for Z-matrices in [6]). Thus the PLCP reduces to one with a P-matrix \(M\) as in Theorem 1.1. However, by
excluding \( q = 0 \), it is possible to have a PLCP\((q + \alpha p, M)\) with isotope complementary solutions for every \( 0 \neq q \geq 0 \) and every \( p \) where \( M \) is not a P-matrix. An example is presented in Section 4.

2. Further Definitions, Notations, and Previous Results

The cone generated by the columns of a matrix \( A \) is denoted by \( \text{Pos}[A] \), i.e., \( \text{Pos}[A] = \{ Ax \mid x \geq 0 \} \). The \( j \)th column of \( A \) is denoted by \( A_j \). If \( A \) is an \( n \times n \) matrix and if for each \( j = 1, 2, \ldots, n \), \( A_j \) is either \( I_j \) (the \( j \)th column of the identity matrix \( I \)) or \( -M_j \) (the \( j \)th column of \( -M \)), then \( \text{Pos}[A] \) is called a complementary cone. The LCP\((q, M)\) has a complementary solution iff \( q \) belongs to some complementary cone. Thus, \( K(M) \) is the union of all complementary cones. A complementary cone whose interior is nonempty is said to be nondegenerate; otherwise, it is said to be degenerate. The interior of \( \text{Pos}[A] \) is denoted by \( \text{int}(\text{Pos}[A]) \). If \( \text{Pos}[A] \) is an \( m \)-dimensional degenerate complementary cone in \( \mathbb{R}^n \), then its relative interior is its interior in \( \mathbb{R}^n \) and is denoted by \( \text{relint}(\text{Pos}[A]) \). The set of complementary cones forms a partition of \( \mathbb{R}^n \) iff their union is \( \mathbb{R}^n \) and they have nonempty interiors which are pairwise disjoint.

For a point \( q \) of a complementary cone \( \text{Pos}[A] \) we set
\[ X(q,A) = \{ x \mid Ax = q, \ x \geq 0 \} \]

For each \( x \in X(q,A) \), the complementary solution of the LCP\((q,M)\) obtained by setting the variables in \((w;z)\) associated with \( A_j \) equal to \( x_j \) and the rest equal to zero is said to be induced by \( \text{Pos}[A] \).

The following theorems will be used to prove the main result.

**Theorem 2.1.** (Cottle and Stone [4]) Let \( \text{Pos}[A] \) be a degenerate complementary cone. For every \( q \) in \( \text{relint}(\text{Pos}[A]) \), the number of complementary solutions of the LCP\((q,M)\) induced by \( \text{Pos}[A] \) is infinite.

**Theorem 2.2** (Eaves [5]) \( M \) is an \( L_1 \)-matrix iff the LCP\((q,M)\) has a unique complementary solution for every \( q \geq 0 \).

**Theorem 2.3.** (Cottle and Dantzig [3]; Eaves [5]) If \( M \) is an \( L_1 \)-matrix, then \( K(M) = R^n \).

**Theorem 2.4.** (Murty [8]; Samelson, Thrall & Wesler [12]) The set of complementary cones forms a partition of \( R^n \) iff \( M \) is a P-matrix.

**Theorem 2.5.** (Cottle [2]) Given the PLCP\((q+\alpha p,M)\) where \( M \) is a P-matrix and \( q \geq 0 \). Then \( z(\alpha) = z(\alpha;q,p) \) is monotone nondecreasing for every \( p \) iff \( (M^*)^{-1}q^* \geq 0 \) for every principal submatrix \( M^* \) of \( M \) and corresponding subvector \( q^* \).
3. **The Main Result**

The proof of the next lemma is straightforward:

**Lemma 3.1.** If \((w;z)\) is a complementary solution of the LCP\((p,M)\), then \((\alpha w;\alpha z)\) is a complementary solution of the LCP\((\alpha p,M)\) for every \(\alpha \geq 0\).

**Lemma 3.2.** Let \(p\) be a point in the relative interior of a degenerate complementary cone \(\text{Pos}[A]\). Then the PLCP\((0+\alpha p,M)\) has a complementary map that is not isotone.

**Proof:** Let \(0 < \alpha_1 < \alpha_2\). Since \(p \in \text{relint}(\text{Pos}[A])\), then \(\alpha_1 p \in \text{relint}(\text{Pos}[A])\). By Theorem 2.1, the number of complementary solutions of the LCP\((\alpha_1 p,M)\) induced by \(\text{Pos}[A]\) is infinite. Let \((w(\alpha_1);z(\alpha_1))\) and \((w'(\alpha_1);z'(\alpha_1))\) be distinct complementary solutions of the LCP\((\alpha_1 p,M)\) induced by \(\text{Pos}[A]\). Then \(z(\alpha_1) \neq z'(\alpha_1)\). Hence, there is an index \(k\) such that \(z_k(\alpha_1) \neq z'_k(\alpha_1)\), say

\[
z_k(\alpha_1) > z'_k(\alpha_1). \tag{2}
\]

Let \(x_j(\alpha_1)\) and \(x'_j(\alpha_1)\) denote the variables in \((w(\alpha_1);z(\alpha_1))\) and \((w'(\alpha_1);z'(\alpha_1))\), respectively, associated with \(A_j\). Then we have

\[
Ax'(\alpha_1) = \alpha_1 p. \tag{3}
\]

From (2) we have
\[ x_k(\alpha_1) = z_k^*(\alpha_1) > z_k^*(\alpha_1) = x_k^*(\alpha_1). \] (4)

Let \((w(\alpha_2); z(\alpha_2))\) be a complementary solution of the LCP\((\alpha_2p, M)\) induced by Pos\([A]\) and let \(x_j(\alpha_2)\) denote the variable in \((w(\alpha_2); z(\alpha_2))\) associated with \(A_j\). Then

\[ Ax(\alpha_2) = \alpha_2p. \] (5)

Define \(\alpha^*\) and \(x(\alpha^*)\) by

\[ \alpha^* = (1-\lambda)\alpha_1 + \lambda\alpha_2, \quad 0 < \lambda < 1, \]

\[ x(\alpha^*) = (1-\lambda)x'(\alpha_1) + \lambda x(\alpha_2). \] (6)

Then \(\alpha_1 < \alpha^* < \alpha_2\), \(x(\alpha^*) \geq 0\), and

\[ Ax(\alpha^*) = (1-\lambda)Ax'(\alpha_1) + \lambda Ax(\alpha_2) \]

\[ = (1-\lambda)\alpha_1p + \lambda\alpha_2p, \quad \text{from (3) and (5)}, \]

\[ = \alpha^*p. \]

Hence, there is a complementary solution \((w(\alpha^*); z(\alpha^*))\) of the LCP\((\alpha^*p, M)\) induced by Pos\([A]\) that is associated with \(x(\alpha^*)\). Now, from (6),

\[ x_k(\alpha^*) = (1-\lambda)x_k'(\alpha_1) + \lambda x_k(\alpha_2) \]

\[ = x_k'(\alpha_1) + \lambda[x_k(\alpha_2) - x_k'(\alpha_1)]. \]

Since \(x_k(\alpha_1) > x_k'(\alpha_1)\), we can choose \(\lambda\) small enough such that

\[ x_k(\alpha_1) > x_k'(\alpha_1) + \lambda[x_k(\alpha_2) - x_k'(\alpha_1)] = x_k(\alpha^*). \]
Thus, 
\[ z_k(\alpha) = x_k^i(\alpha) > x_k^i(\alpha') = z_k(\alpha') \]
and this defines a complementary map that is not isotone. □

Lemma 3.3. If the PLCP(0+αp, M) has isotone complementary solutions for every \( p \), then every complementary cone is nondegenerate.

Proof: Case 1. \( n = 1 \). If there is a degenerate complementary cone, then \( M = M_{11} = 0 \). Let \( p > 0 \). For \( \alpha = 0 \), \( (0; z) \) is a complementary solution of the LCP(0p, M), where \( z > 0 \). For \( \alpha = 1 \), \( (p; 0) \) is a complementary solution of the LCP(1p, M), contrary to isotonicity.

Case 2. \( n > 1 \). If there is a degenerate complementary cone, then there is at least one with a nonempty relative interior. To prove this, we note that every degenerate complementary cone must have at least one column from \( -M \) since \( \text{Pos}[I] \) is nondegenerate. If \( -M \) has no zero column, then every degenerate complementary cone must have dimension \( m \geq 1 \); hence, its relative interior is nonempty. If \( -M \) has a zero column, say \( -M_j = 0 \), then \( \text{Pos}[I_1, \ldots I_{j-1}, -M_j, I_{j+1}, \ldots I_n] \) is degenerate and of dimension \( n-1 \) and has a nonempty relative interior. By Lemma 3.2, there is a complementary map that is not isotone contrary to the hypothesis. □
Lemma 3.4. If the PLCP\((0+\alpha p, M)\) has isotone complementary solutions for every \(p\), then \(M\) is an \(L_\ast\)-matrix.

**Proof.** Let \(p \geq 0\). We show that the LCP\((p,M)\) has a unique complementary solution. Since \(p \geq 0\), then \((p;0)\) is a complementary solution of the LCP\((p,M)\). If the LCP\((p,M)\) has another complementary solution \((w;z)\), then \(0 \preceq z \geq 0\). For \(\alpha > 1\), \(\alpha p \geq 0\); hence, \((\alpha p;0)\) is a complementary solution of the LCP\((\alpha p,M)\) contrary to isotonicity. Thus, the LCP\((p,M)\) has a unique complementary solution for each \(p \geq 0\). By Theorem 2.2, \(M\) is an \(L_\ast\)-matrix. \(\square\)

Theorem 3.1. The PLCP\((0+\alpha p, M)\) has isotone complementary solutions for every \(p\) iff \(M\) is a P-matrix.

**Proof.** \((\Rightarrow)\) By Lemma 3.4, \(M\) is an \(L_\ast\)-matrix and so, by Theorem 2.3, \(K(M) = \mathbb{R}^n\). Moreover, by Lemma 3.3, all the complementary cones are nondegenerate. We show that the complementary cones have pairwise disjoint interiors. Suppose not. Let Pos[A] and Pos[B] be distinct complementary cones whose interiors have a nonempty intersection and let

\[ p \in \text{int}(\text{Pos}[A]) \cap \text{int}(\text{Pos}[B]).\]

Since Pos[A] is not identical to Pos[B], then there is a \(j\) such that \(A_j \neq B_j\), say
$A_j = -M_j$ and $B_j = I_j$.

Let $(w^A(1); z^A(1))$ and $(w^B(1); z^B(1))$ denote the complementary solutions of the LCP($p, M$) induced by Pos[$A$] and Pos[$B$], respectively. Since $p$ is interior to both pos[$A$] and pos[$B$], we must have

$$z^A_j(1) > 0,$$

$$z^B_j(1) = 0 \quad \text{(since } w^B_j(1) > 0).$$

Let $\alpha > 1$ and let $(w^A(\alpha); z^A(\alpha))$ and $(w^B(\alpha); z^B(\alpha))$ be the complementary solutions induced by Pos[$A$] and Pos[$B$], respectively, of the LCP($\alpha p, M$). Then, by Lemma 3.1,

$$z^A_j(\alpha) = \alpha z^A_j(1) > 0$$

$$z^B_j(\alpha) = \alpha z^B_j(1) = 0.$$  

Conditions (7) and (10) violate isotonicity. Thus, the complementary cones form a partition of $\mathbb{R}^n$. By Theorem 2.4, $M$ is a P-matrix.

(→) If $M$ is a P-matrix, then isotonicity coincides with monotonicity and the conclusion follows from Theorem 2.5 with $q = 0$. □

4. Conclusion

If the PLCP($q+\alpha p, M$) has isotone complementary solutions for every $q \geq 0$ and every $p$, then, in particular, it must have isotone complementary
solutions for \( q = 0 \) and every \( p \); hence, Theorem 3.1 shows that \( M \) must be a \( P \)-matrix. Thus, the PLCP is reduced to Cottle's [2] PLCP with a \( P \)-matrix \( M \). As we have shown, the crucial point is isotonicity for \( q = 0 \) and every \( p \). To look for other possibilities, we have to exclude \( q = 0 \). For example, consider the following matrix in [2]:

\[
M = \begin{bmatrix}
  1 & -1 \\
 -1 & 1 
\end{bmatrix}.
\]

The isotonicity property for every \( 0 \leq q \leq 0 \) and every \( p \) can be seen from the complementary cones shown in Fig. 1.

For example, let \( q = [2 \ 1] \) and \( p = [-1 \ -1] \). In this case, we have \( T = [0, 3/2] \). For \( 0 \leq \alpha \leq 1 \), the complementary solution is unique for each \( \alpha \) and

\[
z_1(\alpha) = 0, \quad z_2(\alpha) = 0.
\]

For \( 1 \leq \alpha < 3/2 \), the complementary solution is unique for each \( \alpha \) and

\[
z_1(\alpha) = 0, \quad z_2(\alpha) \text{ increases with } \alpha; \ z_2(\alpha) \rightarrow 1/2 \text{ as } \alpha \rightarrow 3/2.
\]

For \( \alpha = 3/2 \), there are an infinite number of complementary solutions:
$z_1(\alpha) = (0, \infty), \quad z_2(\alpha) = [1/2, \infty)$.

The graphs of $z_1(\alpha)$ and $z_2(\alpha)$ are shown in Fig. 2 which clearly shows the isotonicity of every complementary map.

We note that $M$ is not a P-matrix. Note, however, that $M$ is a Z-matrix. Thus, Z-matrices may still be important in the PLCP$(q+\alpha_p, M)$ where $0 \leq q \leq 0$.

References


References


\[ Z_1(\alpha) \]

\[ Z_2(\alpha) \]

\[ Z_1 \]

\[ Z_2 \]

\[ \frac{1}{2} \]

\[ \frac{3}{2} \]

\[ \frac{3}{2} \]

\[ \alpha \]

\[ \alpha \]