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On Non-equilibrium in Some Steady States

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Abstract

Non-equilibrium is shown for a steady state in Grandmont and Younes (1973) and a steady state in McCallum (1984). The reason for non-equilibrium, which is the same in both cases, can be stated as a general proposition.

1. Introduction

Grandmont and Younes (1973) have analyzed a model of an exchange economy where the stock of money is decreased at the same rate as the common rate of time preference, and they claim that there is a corresponding steady-state equilibrium which is Pareto efficient. In an aggregative model, let a steady state be defined in terms of a constant stock of government bonds or, alternatively, a constant bond-financed budget deficit. McCallum (1984) claims that if the deficit takes account of current interest payments, there is a corresponding steady-state equilibrium with no price inflation. These two well-known papers have been cited with approbation in recent surveys; see Woodford (1990, p. 1085) and Seater (1993, p. 154). In this paper it will be shown that the steady state in each case is in fact not an equilibrium. It will also be seen that the reason for non-equilibrium is the same in both cases.

2. The Grandmont-Younes Model

Assume that each trader i in an exchange economy has an endowment of commodities given by the vector $w_i \geqq 0$, constant in every period $t \geqq 0$, and he wishes to maximize

$$\sum_{t=0}^{\infty} \delta_i^t u_i(c_i(t)) \tag{2.1}$$

where u_i is his utility function, c_i is his consumption, and his rate of time preference $\varsigma_i = (1 - \delta_i)/\delta_i$, $0 < \delta_i < 1$. With fiat money required as medium of exchange, let $M(t) = \alpha M(t-1)$, $0 < \alpha \leqq 1$, be the stock of money in period t. This is effected by collecting from person i, at the beginning of t, the lump-sum amount

$$a_i(t) = \theta_i(\alpha - 1)M(t - 1) = \theta_i(\alpha - 1)\alpha^t M \leqq 0 \tag{2.2}$$

where $M = M(-1)$, $\theta_i > 0$ and $\Sigma\theta_i = 1$. (If $\alpha = 1$, the money supply is constant; the case of interest will have $\alpha < 1$.) The individual trader then has the budget constraint

$$p(t)\cdot c_i(t) + m_i(t) \leqq p(t)\cdot w_i + m_i(t-1) + a_i(t) \tag{2.3}$$

where the price vector $p > 0$ and $m_i(t-1)$ is one's cash holding at the end of $t-1$. Let $G_i(p(t), c_i(t))$ be the minimum amount of cash needed to have $c_i(t)$, given $p(t)$, so the transactions constraint is

$$G_i(p(t), c_i(t)) \leqq m_i(t-1) + a_i(t). \tag{2.4}$$

It is assumed that G_i is convex in c_i and homogeneous of degree one in p.

A steady-state equilibrium is defined by $(p^*, (c_i^*, m_i^*))$ if: $\Sigma(c_i^* - w_i) = 0$; $\Sigma m_i^* = M$; and for every i, the program $(\bar{c}(t), \bar{m}(t))$, where $\bar{c}(t) = c_i^*$ and $\bar{m}(t) = \alpha^{t+1}m_i^*$, solves the problem of maximizing $\Sigma_{t=0}^{\infty} \delta_i^t u_i(c(t))$ subject to

$$\alpha^{t+1}p^*\cdot(c(t) - w_i) + m(t) \leq m(t-1) + a_i(t) \tag{2.5}$$

$$G_i(\alpha^{t+1}p^*, c(t)) \leq m(t-1) + a_i(t) \tag{2.6}$$

and $c(t) \geq 0$, $m(t) \geq 0$ for $t \geq 0$, with $m(-1) = m_i^*$.

Grandmont and Younes show that a solution for i's maximization problem requires that the Lagrange multipliers $\lambda_i > 0$ and $\pi_i \geq 0$, associated with the budget constraint (2.5) and the transactions constraint (2.6) respectively, be such that

$$\lambda_i \geq (\lambda_i + \pi_i)\delta_i/\alpha \tag{2.7}$$

To see this, divide (2.5) and (2.6) by α^{t+1} to get the steady-state single-period conditions

$$p^*\cdot(c(t) - w_i) \leq (m_i^* - \theta_i M)(1 - \alpha)/\alpha$$

$$G_i(p^*, c(t)) \leq m_i^*/\alpha - \theta_i M(1 - \alpha)/\alpha.$$

Suppose i shifts from $(\alpha^{t+1}m_i^*)$ to a higher $(\alpha^{t+1}(m_i^* + dm_i))$ for $t \geq 0$, and $(\alpha^{t+1}m_i^*)$ is steady-state optimal. The marginal loss of utility in period 0 is $\lambda_i dm_i$ and the undiscounted marginal gain in every period $t \geq 1$ is $\lambda_i dm_i(1 - \alpha)/\alpha + \pi_i dm_i/\alpha$.

Recalling that his rate of time preference $\zeta_i = (1 - \delta_i)/\delta_i$, the total discounted net gain is thus

$$[-\lambda_i + \lambda_i \frac{(1-\alpha)}{\alpha} \frac{\delta_i}{(1-\delta_i)} + \frac{\pi_i}{\alpha} \frac{\delta_i}{(1-\delta_i)}] \, dmi \leq 0 \tag{2.8}$$

which gives (2.7).

Suppose $\delta_i = \delta$ for all i. There is no steady-state equilibrium if $\alpha < \delta$, for (2.7) is violated. If $\alpha > \delta$, Grandmont and Younes show that the steady-state equilibrium is Pareto inefficient. It is the third possibility where $\alpha = \delta$ that concerns us.

Consider the case where $\alpha = \delta$. This implies $\pi_i = 0$ in (2.7), which means that transactions constraints are not binding. Grandmont and Younes (1973, p. 159) correctly point out that "it is not worthwhile to decrease [cash] balances ... from $(\alpha^{t+1}m_i^*)$ to $(\alpha^{t+1}(m_i^* - dm_i))$ for $t \geq 0$" since, looking at (2.8), the net gain would be zero. However, an individual trader is not confined to steady-state solutions of maximizing (2.1). He can choose to draw down his cash balance to $m_i^* - dm_i$ in period 0 and revert to $(\alpha^{t+1}m_i^*)$ for $t \geq 1$. He thereby gains $\lambda_i dm_i$ in period 0 and loses $\lambda_i dm_i(1 - \alpha)\delta/\alpha$ in period 1 for a net gain of $\alpha\lambda_i dm_i$, and therefore the steady state with $\pi_i = 0$ cannot be an equilibrium.

3. The McCallum Model

This is an "aggregative ... equilibrium model" (McCallum, 1984, p. 124) where p is now the price of the single product which is added to the capital stock k if not consumed. The representative individual wishes to maximize

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t, m_t) \tag{3.1}$$

where $m_t = M_t/p_t$ is now in real terms with M_t in per capita terms. His budget constraint is

$$\begin{aligned} f(k_t) + v_t = c_t + (1 + \pi_t)m_{t+1} - m_t + (1 + r_t)^{-1} b_{t+1} \\ - b_t + k_{t+1} - k_t \end{aligned} \tag{3.2}$$

where f is his production function, v denotes lump-sum transfers received net of taxes, $\pi_t = (p_{t+1} - p_t)/p_t$ is the inflation rate, r is the real rate of return on bonds given by $1 + r_t = (1 + R_t)/(1 + \pi_t)$ where $R_t = (1 - Q_t)/Q_t$ with Q the money price of a bond redeemed for one unit of money after being held one time-period, and $b_t = B_t/p_t$ where $B_t \geq 0$ is the number of bonds at the start of t which were held during $t - 1$.

Under the usual assumptions on u and f (Sidrauski, 1967), one would have the following optimality conditions:

$$u_1(c_t, m_t) - \lambda_t = 0 \tag{3.3}$$

$$\delta[u_2(c_{t+1}, m_{t+1}) + \lambda_{t+1}] - \lambda_t(1 + \pi_t) = 0 \tag{3.4}$$

$$\delta\lambda_{t+1}[f'(k_{t+1}) + 1] - \lambda_t = 0 \tag{3.5}$$

$$\delta\lambda_{t+1} - \lambda_t(1 + r_t)^{-1} \leq 0 \tag{3.6a}$$

$$[\delta\lambda_{t+1} - \lambda_t(1 + r_t)^{-1}]b_{t+1} = 0 \tag{3.6b}$$

$$\lim_{t\to\infty} m_{t+1}\delta^{t-1}\lambda_t(1+\pi_t) = 0 \tag{3.7}$$

$$\lim_{t\to\infty} k_{t+1}\delta^{t-1}\lambda_t = 0 \tag{3.8}$$

$$\lim_{t\to\infty} b_{t+1}\delta^{t-1}\lambda_t(1+r_t)^{-1} = 0 \tag{3.9}$$

and $c_t > 0$, $m_{t+1} > 0$, $b_{t+1} \gtreqless 0$, $t = 1, 2, \ldots$; $\lambda > 0$ is the Lagrange multiplier associated with the budget constraint. It is known that (3.2)-(3.9) are jointly sufficient for optimality and (3.2)-(3.6) are necessary, not counting the inequality constraints on c, m and b.

The government's budget identity per capita can be written as

$$M_{t+1} - M_t + Q_t B_{t+1} - B_t = p_t(g_t + v_t) \tag{3.10}$$

where g is government purchase of output. Given the time-paths of the policy variables M, g and v, the paths of all the other variables are determined by (3.2)-(3.6), (3.10) and the definitions of the terms involved. McCallum (1984, p. 130) shows that there is no "zero-inflation equilibrium in which a permanently maintained positive deficit of $g_t + v_t = d$ is financed entirely by bond sales." The case that concerns us is where the deficit is defined as usual to include current interest payments. Is there a corresponding zero-inflation equilibrium?

McCallum's argument is as follows. Let $\overline{B}_t = B_t/(1 + R_{t-1})$ be the issue value of bonds at t so that (3.10) becomes

$$M_{t+1} - M_t + \overline{B}_{t+1} - \overline{B}_t = p_t(g_t + v_t) + R_{t-1}\overline{B}_t. \tag{3.11}$$

Suppose the time-paths of M, $\bar{g}$ and v are chosen so that M and g are constant and in real terms the right-hand side of (3.11) is equal to a constant $\bar{d} > 0$, in which case

$$(\bar{B}_{t+1} - \bar{B}_t)/p_t = \bar{d}. \tag{3.12}$$

The time-paths of all the variables are thus determined. Consider the possibility that one has constant c, k and p, which implies $\pi = 0$ and $m = \text{const}$. Then $\lambda = \text{const}$ in (3.3), and with $b_{t+1} > 0$ from (3.12), $r = \text{const}$ in (3.6b), so (3.12) reduces to

$$b_{t+1} - b_t = (1 + r)\bar{d} \tag{3.13}$$

which gives

$$b_{t+1} = b_1 + (1 + r)t\bar{d} \tag{3.14}$$

and consequently the transversality condition (3.9) is satisfied. Since it can be quickly checked that (3.2)-(3.8) are also satisfied in this steady state with a constant $\bar{d}$, there would seem to be an affirmative answer to the question posed.

However, the above argument rests on (3.12), which in effect requires the representative individual to buy the bonds $b_{t+1} > 0$ for all $t \geqq 1$. But the fact is that one can choose $b_{t+1} = 0$ at some $t \geqq 1$ to increase his consumption without changing k or m, and therefore such a steady state cannot be an equilibrium.

It is equally clear that a steady state with a constant $b > 0$, which might seem to obtain where $b_1 > 0$ and $\bar{d} = 0$, cannot be an equilibrium either.

4. Concluding Remarks

We have seen that in the Grandmont-Younes model, where money does not appear in the traders' utility functions but the transactions constraint is satisfied as a strict inequality, and in the McCallum model, where bonds are not in the utility function but the bond nonnegativity constraint is satisfied as a strict inequality, one does not have an equilibrium in the corresponding steady states. Although the settings are dissimilar, they share one feature that accounts for non-equilibrium which can be stated in the form of the following:

Proposition. Let x be a financial asset subject to a constraint $x_t \geq \Phi(\cdot)$ for all t in an infinite-horizon discounted utility maximization problem. If x is not an argument of the utility function and the constraint is satisfied as a strict inequality in a steady state, the latter is not an equilibrium.

The reason is simply that a financial asset is held because it can be exchanged at a later time for something that gives utility. By such an exchange, utility is higher, and therefore a steady state where more of the asset is held than is necessary cannot be optimal hence not an equilibrium.

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