A Characterization of Q-Matrices

by

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A Characterization of $q$-Matrices

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Abstract. Let $K(M)$ denote the set of all $q \in \mathbb{R}^n$ such that the linear complementarity problem $LCP(q, M)$ has a complementary solution. We show that (a) $M$ is an $S$-matrix iff there is a $q^0 \in K(M)$ such that $q^0 < 0$ and (b) $M$ is a $Q$-matrix iff $M$ is a $Q_0$-matrix and an $S$-matrix.

Key Words. Linear complementarity problem, matrices, separating hyperplane theorem.

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1. Introduction.

Given an n x n matrix \( M \) with real entries and a vector \( q \in \mathbb{R}^n \), the linear complementarity problem \( \text{LCP}(q, M) \) is the problem of finding vectors \( w, z \in \mathbb{R}^n \) such that

\[
\begin{align*}
Iw - Mz &= q \\
0 \leq w, \\n0 \leq z \\
w^T z &= 0.
\end{align*}
\]

A pair \((w; z)\) is called a feasible solution if it satisfies (1) and (2); it is called a complementary solution if it satisfies (1), (2), and (3). The set of all \( q \in \mathbb{R}^n \) for which the \( \text{LCP}(q, M) \) has a complementary solution is denoted by \( K(M) \).

The linear complementarity problem arises in mathematical programming (Eaves (Ref. 1)), game theory (Lemke (Ref. 2)), and economic equilibrium theory (Mathiesen (Ref. 3)).

For a certain class of matrices \( M \), the existence of a feasible solution implies the existence of a complementary solution. Following Cottle (Ref. 4) we shall call these matrices the \( Q_0 \)-matrices. (These matrices are also called K-matrices). They include the copositive plus matrices (which include the positive semidefinite matrices) (Lemke, Ref. 2), adequate matrices (Ingleton, Ref. 5), and E-matrices (Chandrasekaran, Ref. 6).
$Q_0$-matrices are characterized by the convexity of $K(M)$ (Eaves, Ref. 7). It follows that $Q$-matrices (those for which $K(M) = \mathbb{R}^n$) are $Q_0$-matrices. These include the $L_\infty$-matrices (Eaves, Ref. 1), $P$-matrices (Murty, Ref. 8), and regular matrices (Karamardian, Ref. 9).

It is natural to ask what additional conditions are required for a $Q_0$-matrix to be a $Q$-matrix. We show that a necessary and sufficient condition for a $Q_0$-matrix $M$ to be a $Q$-matrix is that $M$ be an $S$-matrix. Thus the intersection of the $Q_0$-matrices and the $S$-matrices consists of the $Q$-matrices. A characterization of $Q$-matrices within the class of $P_0$-matrices is given in Anagagic and Cottle (Ref. 10) where it is shown that among the $P_0$-matrices the $Q$-matrices are precisely the regular matrices.

2. Notations and Preliminaries

Let $\text{Pos}[A]$ denote the cone generated by the column vectors of a matrix $A$, i.e.,

$$\text{Pos}[A] = \{q: q = Ax, x \geq 0\}.$$  

Consider the complementarity matrix $[I, -M]$. If $A$ is a matrix whose $j$th column $A_j$ is either $I_j$ (the $j$th column of $I$) or $-M_j$ (the $j$th column of $-M$), then $\text{Pos}[A]$ is called a complementary cone. The $\text{LCP}(q, M)$ has a complementary solution iff $q$ belongs to a complementary cone. Thus, $K(M)$ is the union of all complementary cones. In
the rest of this note, $M$ is an $n \times n$ matrix. The interior of a set $C$ is denoted by $\text{int}(C)$.

The following results will be used in the proof of the main theorem.

**Lemma 1.** (Paves, Ref. 7) The following statements are equivalent: (i) $M$ is a $Q_o$-matrix;

(ii) $K(M)$ is convex;

(iii) $K(M) = \text{Pos}[I,-M]$.

**Lemma 2.** Let $C_1$ and $C_2$ be nonempty disjoint convex sets in $\mathbb{R}^n$. Then there exists a hyperplane that separates them.

Proof: Mangasarian (Ref. 11).

**Definition 1.** $M$ is an $S$-matrix iff there exists a $z^0 \geq 0$ such that $Mz^0 > 0$.

**Remark 1.** In literature, $S$-matrices are defined for any rectangular matrix. The next lemma characterizes square $S$-matrices in terms of the linear complementarity problem.

**Lemma 3.** $M$ is an $S$-matrix iff there exists a $q^0 \in K(M)$ such that $q^0 < 0$. 


Proof: (.) If \( M \) is an \( S \)-matrix, then there is a \( z^0 \geq 0 \) such that \( Mz^0 > 0 \). Define \( q^0 = -Mz^0 \). Then \( q^0 < 0 \) and \( q^0 \in \text{Pos}[-M] \subseteq K(M) \).

(. ) If \( M \) is not an \( S \)-matrix, then the system \( Mz > 0, \ z \geq 0 \) has no solution, i.e., the system \( -Mz < 0, \ z \geq 0 \) has no solution. This implies that the complementary cone \( \text{Pos}[-M] \) has no point in the interior of the nonpositive orthant \( \text{Pos}[-I] \), i.e.,

\[
\text{Pos}[-M] \cap \text{int}(\text{Pos}[-I]) = \emptyset.
\]

By Lemma 2, there exists a hyperplane \( H \) separating \( \text{Pos}[-M] \) and \( \text{int}(\text{Pos}[-I]) \); hence, \( \text{Pos}[-M] \) and \( \text{int}(\text{Pos}[I]) \) are contained in the same closed half-space \( H' \). Since the closure of \( \text{int}(\text{Pos}[I]) \) is \( \text{Pos}[I] \), then \( \text{Pos}[I] \subseteq H' \). Hence, \( \text{Pos}[-M] \) and \( \text{Pos}[I] \) are contained in \( H' \) which implies that all the complementary cones and, therefore, \( K(M) \), are contained in \( H' \). Hence, \( K(M) \) has no point in the interior of \( \text{Pos}[-I] \), contrary to the hypothesis.

3. The Main Result

Theorem 1. \( M \) is a \( Q \)-matrix iff \( M \) is a \( Q_0 \)-matrix and an \( S \)-matrix.

Proof: (.) Since \( M \) is a \( Q \)-matrix, then \( K(M) = R^n \); hence, it is a \( Q_0 \)-matrix by Lemma 1 and an \( S \)-matrix by Lemma 3.
Let \( q \in \mathbb{R}^n \). We wish to show that \( q \in K(M) \). Since \( M \) is an \( s \)-matrix, then, by Lemma 3, there is a \( q^0 \in K(M) \) such that \( q^0 \leq 0 \). We have
\[
q^0 = \sum_{j=1}^{n} q^0_j I_j \quad (4)
\]
and
\[
q = \sum_{j=1}^{n} q_j I_j. \quad (5)
\]
Now,
\[
q = \lambda q^0 + q - \lambda q^0, \quad (\lambda > 0) \quad (6)
\]
Since \( q^0_j < 0 \) \((j=1,2,\ldots,n)\), we can choose \( \lambda \) large enough such that \( (q_j - \lambda q^0_j) > 0 \) \((j=1,2,\ldots,n)\). Hence, \( q \) can be expressed as a nonnegative linear combination of points in \( K(M) \). Since \( M \) is a \( Q_0 \)-matrix, then \( K(M) \) is a convex cone (Lemma 1); hence, \( q \in K(M) \). It follows that \( K(M) = \mathbb{R}^n \) and \( M \) is a \( Q \)-matrix.

\[\Box\]

4. A Remark on \( P \)-matrices

Among the \( Q \)-matrices, the \( P \)-matrices have been widely studied. It is well-known that the \( LCP(q,M) \) has a unique complementary solution for each \( q \in \mathbb{R}^n \) iff \( M \) is a \( P \)-matrix (Ref. 8). It is natural to ask what class of matrices \( M \) has the property that the \( LCP(q,M) \) has a unique complementary solution for each \( q \in K(M) \).
Suppose that the $LCP(q,M)$ has a unique complementary solution for each $q \in K(M)$. For each $q \geq 0$, the $LCP(q,M)$ has a complementary solution $((w,z) = (q;0))$ which, by hypothesis, is unique. Eaves (Ref. 1) showed that in this case, $M$ is an $L$-matrix. ($M$ is an $L$-matrix iff for every $z \in R^n$ such that $0 \neq z \geq 0$, there is a $j$ such that $z_j > 0$ and $(Mz)_j > 0$.) The class of $L$-matrices coincides with the class of matrices $M$, defined by Cottle and Dantzig (Ref. 12), having the property that for every principal submatrix $M_{ij}$ of $M$, the system $M_{ij}z_j \leq 0$, $0 \neq z_j \geq 0$ has no solution. Cottle and Dantzig showed that this class of matrices are $Q$-matrices. Therefore, the $LCP(q,M)$ must have $K(M) = R^n$ and $M$ is a $P$-matrix. We thus have the following result.

Theorem 2. If the $LCP(q,M)$ has a unique complementary solution for each $q \in K(M)$, then $K(M) = R^n$ and $M$ is a $P$-matrix.
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