THE LINEAR COMPLEMENTARITY PROBLEM: SOLVING LINEAR AND CONVEX QUADRATIC PROGRAMMING PROBLEMS BY LEMKE'S ALGORITHM

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1. THE LINEAR COMPLEMENTARITY PROBLEM

1.1 Statement of the Problem

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and let $\mathbb{R}^{n \times n}$ be the set of nxn matrices with real entries. Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem, denoted by $(q,M)$, is the following problem:

Find $w, z \in \mathbb{R}^n$ 

Subject to $w = Mz + q$ \hspace{1cm} (1)

$w \geq 0, \quad z \geq 0$ \hspace{1cm} (2)

$w^t z = 0$ \hspace{1cm} (3)

where $w^t$ denotes the transpose of $w$. The linear complementarity problem $(q,M)$ is said to be of order $n$.

Since $w \geq 0$ and $z \geq 0$, the constraint $w^t z = \sum_{j=1}^{n} w_j z_j = 0$ implies that $w_j z_j = 0$ for each $j = 1, 2, \ldots, n$. The variables $w_j$ and $z_j$ are said to be complementary. Any pair $(w;z)$ that satisfies (1), (2), and (3) is called a complementary solution and the set of all complementary solutions of $(q,M)$ is denoted by $C(q,M)$. Note that if $q \geq 0$, the linear complementarity problem $(q,M)$ always has a complementary solution given by $(w;z) = (q;0)$.

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The linear complementarity problem has been shown to be a unified form of problems in diverse fields of study such as mathematical programming, game theory, mechanics, and economics.

Cottle and Dantzig [1] and Lemke [6] showed that linear programming problems, convex quadratic programming problems, and finding Nash equilibrium points of bimatrix games can be formulated as linear complementarity problems. Ingleton [5] describes Lagrange's equation of motion for the initial velocities under applied impulses of a dynamical system subject to smooth unilateral constraints and shows how this equation can be reduced to the form of the linear complementarity problem. Dantzig and Manne [2] analyzed the invariant capital stock problem from the viewpoint of Lemke's linear complementarity algorithm.

In the next two sections the linear programming problem and the convex quadratic programming problem will be formulated as linear complementarity problems. These formulations are due to Cottle and Dantzig [1].

1.2 The Linear Programming Problem

Consider the following pair of dual linear programming problems:

Primal: \[ \text{Min } c^T x \]
\[ \text{s.t. } Ax \geq b \]
\[ x \geq 0 \]

Dual: \[ \text{Max } b^T y \]
\[ \text{s.t. } y^T A \leq c \]
\[ y \geq 0 \]
Dual: \[ \text{Max } b^T y \]
\[ \text{s.t. } A^T y \leq c \]
\[ y \geq 0 \] (D)

The following theorem gives a necessary and sufficient condition for the primal and the dual problems to have optimal solutions:

1.1.1 Theorem. (Weak Theorem of Complementary Slackness)

Given the pair of dual linear programming problems \( P \) and \( D \), a necessary and sufficient condition for \( \bar{x} \) and \( \bar{y} \) to be optimal is that they satisfy the equations

\[ y^T (Ax - b) = 0 \]
\[ (c - A^T y)^T x = 0 \].

Proof: See Simmonard [9]. \( \square \)

If we let \( u = -A^T y + c \) \( \quad (4) \)

and \( v = Ax - b \), \( \quad (5) \)

then \( u \geq 0 \) and \( v \geq 0 \) and the equations in Theorem 1 become

\[ y^T v + x^T u = 0 \]. \( \quad (6) \)
We can rewrite conditions (4), (5), and (6) as follows:

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = \begin{bmatrix}
  0 & -A^t \\
  A & 0
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix} + \begin{bmatrix}
  c \\
  -b
\end{bmatrix} \tag{7}
\]

\[x^t u + y^t v = 0, \tag{8}\]

where \( x \geq 0, y \geq 0, u \geq 0, \) and \( v \geq 0. \) \tag{9}\]

The following identifications express conditions (7), (8), and (9) as a linear complementarity problem:

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}, \quad M = \begin{bmatrix}
  0 & -A^t \\
  A & 0
\end{bmatrix}, \quad w = \begin{bmatrix}
  x \\
  y
\end{bmatrix}, \quad q = \begin{bmatrix}
  c \\
  -b
\end{bmatrix}.
\]

1.3 The Convex Quadratic Programming Problem

1.3.1 Definition. Let \( C \) be a convex subset of \( \mathbb{R}^n \). A function \( f \) defined on the set \( C \) is convex iff for each pair \( x, y \in C \) and \( \lambda \in [0,1] \),

\[ f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y). \tag{3}\]

The following is the usual formulation of the convex quadratic programming problem:

\[
\begin{align*}
\text{Min} & \quad c^t x + \frac{1}{2} x^t Q x \\
\text{Subject to} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]
where $c \in \mathbb{R}^n$, $Q$ is a symmetric $n \times n$ matrix, $A$ is an $m \times n$ matrix, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and the objective function $c^t x + \frac{1}{2} x^t Q x$ is a convex function.

Remark: The requirement that $Q$ be symmetric is not restrictive. For if $Q$ were not symmetric, replace $Q$ by $\tilde{Q} = \frac{1}{2}(Q + Q^t)$, which is symmetric. One can easily show that $x^t Q x = x^t \tilde{Q} x$.

In order to formulate the convex quadratic programming problem as a linear complementarity problem, we need a well-known theorem in mathematical programming namely, the Kuhn-Tucker Theorem. The statement of the theorem is taken from Zangwill [10].

1.3.2 Theorem (Kuhn-Tucker): Consider the nonlinear programming problem

\[
\text{Max } f(x),
\]

Subject to $g_i(x) \geq 0$, $i = 1, \ldots, m$

where all functions are differentiable. Let $\bar{x}$ be an optimal solution and assume the constraint qualification\(^1\) holds.

\(^1\)The constraint qualification is a restriction on the constraint functions. It precludes certain irregularities (e.g., a cusp) on the boundary of the feasible region which would invalidate the Kuhn-Tucker conditions should the optimal solution occur there. For a discussion of the constraint qualification, see Zangwill[10] or Mangasarian [7].
Then there exist multipliers \( \lambda_i \geq 0, i = 1, 2, \ldots, m \) such that

\[
\begin{align*}
(1) & \quad \lambda_i g_i(x) = 0, \ i = 1, 2, \ldots, m \\
(2) & \quad \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0.
\end{align*}
\]

Remarks: 1. If the functions \( f(x), g_i(x) (i = 1, 2, \ldots, m) \) are concave, then the Kuhn-Tucker conditions (10) and (11) are also sufficient. (See Zangwill [10]).

2. One can show that if the constraints are linear, then the constraint qualification holds. (See Zangwill [10]).

The Kuhn-Tucker conditions for the convex quadratic programming problem is obtained by first rewriting the problem into the following form:

\[
\begin{align*}
\text{Max} & \quad -c^T x - \frac{1}{2}x^T Q x \\
\text{Subject to} & \quad g_1(x) = A_1 x - b_1 \geq 0 \\
& \quad \vdots \\
& \quad g_m(x) = A_m x - b_m \geq 0 \\
& \quad g_{m+1}(x) = x_1 \geq 0 \\
& \quad \vdots \\
& \quad g_{m+n}(x) = x_n \geq 0
\end{align*}
\]

\( ^2 \)A function h defined on a convex set C is concave iff \( -h \) is convex.
where $A_i$ is the $i$th row of $A$. Note that the objective function $-c^T x - \frac{1}{2}x^T Q x$ is concave and the constraint functions, being linear, are concave. Hence, a necessary and sufficient condition for $\bar{x}$ to be optimal is that there exist multipliers $\bar{y}_i \geq 0$ ($i = 1, 2, \ldots, m$) and $\bar{u}_i \geq 0$ ($i = 1, 2, \ldots, n$) such that

$$\bar{y}_i (A_i, \bar{x} - b_i) = 0, \quad i = 1, 2, \ldots, m \quad (12)$$

$$\bar{u}_i x_i = 0, \quad i = 1, 2, \ldots, n \quad (13)$$

$$-c - Q \bar{x} + \sum_{i=1}^{m} \bar{y}_i A_i^T + \sum_{i=1}^{n} \bar{u}_i I_i = 0 \quad (14)$$

where $I_i$ is the $i$th row of the identity matrix $I$. The conditions (12), (13), and (14) can be written as

$$\bar{y}^T (A \bar{x} - b) = 0$$

$$\bar{u}^T \bar{x} = 0$$

$$-c - Q \bar{x} + A^T \bar{y} + \bar{u} = 0.$$  

Therefore, a necessary and sufficient condition for $\bar{x}$ to be an optimal solution of the convex quadratic programming problem is that there exist $\bar{y} \geq 0$, $\bar{u} \geq 0$ and $\bar{v} = A \bar{x} - b$ that solve the system of equations

$$u = Q x - A^T y + c \quad (15)$$

$$v = A x - b \quad (16)$$

$$u^T x + y^T v = 0 \quad (17)$$
By setting \( w = \begin{bmatrix} u \\ v \end{bmatrix}, \ z = \begin{bmatrix} x \\ y \end{bmatrix}, \ M = \begin{bmatrix} Q & -A^t \\ A & 0 \end{bmatrix}, \)

and \( q = \begin{bmatrix} c \\ -b \end{bmatrix}, \) the system (15), (16), (17) has the form of a linear complementarity problem.
2. THE GEOMETRY OF THE LINEAR COMPLEMENTARITY PROBLEM

The linear complementarity problem admits of a geometric interpretation that is intuitively appealing (Murty [8]). First, we rewrite the linear complementarity problem \((q, M)\) into the following form:

Find \(w, z \in \mathbb{R}^n\)

Subject to

\[
\begin{bmatrix}
I & -M
\end{bmatrix}
\begin{bmatrix}
w \\
z
\end{bmatrix} = q
\]

\(w \geq 0, z \geq 0\)

\(w^T z = 0\). \hspace{1cm} (18) \hspace{1cm} (19) \hspace{1cm} (20)

2.1 Complementary Cones

Notation. The \(j\)th column of a matrix \(A\) is denoted by \(A_{\cdot j}\).

2.1.1 Definition. Given an \(m \times n\) matrix \(A\), the cone generated by the columns of \(A\), denoted by \(\text{Pos}[A]\), is the set of all non-negative linear combinations of the columns of \(A\), i.e.,

\[
\text{Pos}[A] = \{u \mid u = \sum_{j=1}^{n} x_j A_{\cdot j}, x_j \geq 0\}.
\]
In matrix notation,

\[ \text{Pos}[A] = \{ u \mid u = Ax, \ x \geq 0 \} \]

Each column \( A_j \) is called a generator of the cone \( \text{Pos}[A] \).

2.1.2 Definitions. Let \( M \in \mathbb{R}^{n \times n} \) and let \( I \) be the \( n \times n \) identity matrix. Consider the matrix \( \begin{bmatrix} I_j & -M_j \end{bmatrix} \). For each \( j = 1, 2, \ldots, n \), the vectors \( I_j \) and \( -M_j \) are called complementary vectors. An \( n \times n \) matrix whose \( j \)th column is either \( I_j \) or \( -M_j \) is called a complementary matrix. The cone generated by a complementary matrix is called a complementary cone.

Example

\[ M = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \]
The complementary matrices are

\[ I = \begin{bmatrix} I_1 & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} I_1 & -M_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix} \]

\[ B = \begin{bmatrix} -M_1 & I_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \]

\[ -M = \begin{bmatrix} -M_1 & -M_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix} \]

The complementary cones are shown in Figure 1.
Note: The vectors $I_{1}, I_{2}, -M_{1}$, and $-M_{2}$ are not drawn to scale. A curved arrow joining two vectors indicate the cone generated by these vectors.

Remarks: 1. If $M$ is an $nxn$ matrix, then there are $2^n$ complementary cones associated with $M$. The set of complementary cones associated with $M$ is denoted by $K(M)$.

2. Whenever a complementary cone is denoted by $\text{Pos}[A]$, it is assumed that $A$ is a complementary matrix. The complementary cone $\text{Pos}[I]$ is simply the non-negative orthant of $\mathbb{R}^n$.

2.2. Complementary Solutions Induced by Complementary Cones

It is clear from the definition of a complementary cone that the linear complementarity problem $(q, M)$ has a complementary solution if and only if $q$ belongs to some complementary cone. Thus, the union of all the complementary cones associated with $M$ is the set of all $q \in \mathbb{R}^n$ such that $(q, M)$ has a complementary solution.

If $q$ is a point of a complementary cone $\text{Pos}[A]$, then

$$q = \sum_{j=1}^{n} x_{j}A_{j} \quad \text{where} \quad x_{j} \geq 0 \quad (j = 1, 2, \ldots, n).$$
A complementary solution \((w;z)\) of \((q,M)\) can be obtained by setting the variable associated with \(A_{.j}\) equal to \(x_j\) for \(j = 1, 2, \ldots, n\) and all other variables in \((w;z)\) equal to zero, i.e.,

\[
\begin{align*}
    w_j &= \begin{cases} 
        x_j & \text{if } A_{.j} = I_{.j}, \ j = 1, 2, \ldots, n \\
        0 & \text{if } A_{.j} = -M_{.j}
    \end{cases} \\
    z_j &= \begin{cases} 
        0 & \text{if } A_{.j} = I_{.j}, \ j = 1, 2, \ldots, n \\
        x_j & \text{if } A_{.j} = -M_{.j}
    \end{cases}
\end{align*}
\]

In this case, \((w;z)\) is referred to as a complementary solution induced by \(\text{Pos}[A]\).

**Example**

\[
M = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}, \quad q = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]
Figure 2

Note that $q$ is a point of $\text{Pos } [I_1, I_2]$ and $\text{Pos } [I_1', -M_2]$. The complementary solution of $(q, M)$ induced by $\text{Pos } [I_1, I_2]$ is given by

$$w_1 = 3 \quad z_1 = 0$$

$$w_2 = 1 \quad z_2 = 0.$$  

The complementary solution of $(q, M)$ induced by $\text{Pos } [I_1', -M_2]$ is easily obtained by solving the following equation:

$$w_1 I_1 + z_2 (-M_2) = q,$$
i.e.
\[ w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \]

We get
\[ w_1 = \frac{10}{3}, \quad z_2 = \frac{1}{3}. \]

The complementary solution induced by Pos \([I_1; -M_2]\) is, therefore, given by
\[ w_1 = \frac{10}{3}, \quad z_1 = 0 \]
\[ w_2 = 0, \quad z_2 = \frac{1}{3}. \]

2.3 Principal Submatrices and Complementary Cones

2.3.1 Notation. Let \( M \in \mathbb{R}^{n \times n} \). Let \( J \) and \( \overline{J} \) be subsets of \( \{1, 2, \ldots, n\} \). \( M_{J\overline{J}} \) denotes the submatrix of \( M \) obtained by deleting the rows of \( M \) corresponding to indices not contained in \( J \) and the columns of \( M \) corresponding to the indices not contained in \( \overline{J} \).

Example:
\[
M = \begin{bmatrix}
1 & 0 & -2 & 4 & 3 \\
0 & 6 & 5 & -3 & 1 \\
8 & 9 & 0 & -4 & 6 \\
7 & 2 & 5 & 1 & 0 \\
1 & 0 & 3 & 1 & 1
\end{bmatrix}
\]
2.3.2 Definitions. Let $M \in \mathbb{R}^{n \times n}$ and let $J \subseteq \{1, 2, \ldots, n\}$ consisting of $r$ elements. The submatrix $M_{JJ}$ of $M$ is called a principal submatrix of order $r$. When $r = n$, $M_{JJ}$ is called a proper principal submatrix. When $J = \{1, 2, \ldots, n\}$, then $M_{JJ} = M$. When $J = \emptyset$, then $M_{\emptyset \emptyset}$ is called the empty principal submatrix.

Remark: If $M \in \mathbb{R}^{n \times n}$, then $M$ has $2^n$ principal submatrices.

The determinant of a principal submatrix is called a principal minor. We adopt the convention that $\det M_{\emptyset \emptyset} = 1$. 

$$J = \{1, 4\}$$
$$\bar{J} = \{2, 4, 5\}$$
$$M_{J\bar{J}} = \begin{bmatrix} 0 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$
$$M_{J\bar{J}} = \begin{bmatrix} 0 & -3 \\ 7 & 1 \end{bmatrix}$$
$$M_{JJ} = \begin{bmatrix} 1 & 4 \\ 7 & 1 \end{bmatrix}$$
$$M_{J\bar{J}} = \begin{bmatrix} 6 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$
2.3.3 Definition. $\overline{M}$ is a principal rearrangement of a square matrix $M$ if there exists a permutation matrix $P$ such that

$$\overline{M} = P^T M P.$$ 

Remarks: 1. Given $M \in \mathbb{R}^{n \times n}$, and a principal submatrix $M_{JJ}$, there exists a principal rearrangement $\overline{M}$ of $M$ such that

$$\overline{M} = \begin{bmatrix} M_{JJ} & M_{\overline{J} \overline{J}} \\ M_{\overline{J}J} & M_{\overline{J}\overline{J}} \end{bmatrix},$$

where $\overline{J} = \{1, 2, \ldots, n\} - J$.

2. If $\overline{M}$ is a principal rearrangement of $M$, then $\det \overline{M} = \det M$.

2.3.4 Definition. Let $M \in \mathbb{R}^{n \times n}$, $N = \{1, 2, \ldots, n\}$. Let $J \subseteq N$ and $\overline{J} = N - J$.

Define

$$A_{\cdot J} = \begin{cases} -M_{\cdot j}, & j \in J \\ I_{\cdot j}, & j \in \overline{J} \end{cases}$$

The matrix $A(J) = [A_{1 \cdot J}, A_{2 \cdot J}, \ldots, A_{n \cdot J}]$ is the complementary matrix associated with the set of indices $J$. 
Define the correspondence

\[ \text{Pos}[A(J)] \leftrightarrow ^{\text{MJJ}}. \]

This is a 1-1 correspondence between the set of complementary cones associated with \( M \) and the set of principal submatrices of \( M \).

2.3.5 Theorem. Let \( ^{\text{MJJ}} \) be a principal submatrix of \( M \) and let \( \text{Pos}[A(J)] \) be its corresponding complementary cone. Then

\[ \det A(J) = (-1)^r \det ^{\text{MJJ}} \]

where \( r \) is the order of \( ^{\text{MJJ}} \).

**Proof:** If \( J = \phi \), \( A(J) = I \)

\[ ^{M_{\phi \phi}} = ^{M_{JJ}} \]

\[ \det I = 1 = (-1)^0 \det ^{M_{\phi \phi}}. \]

If \( J \neq \phi \), then by a principal rearrangement, the matrix \( A(J) \) can be transformed into

\[ A(J)^* = \begin{bmatrix} ^{-M_{JJ}} & 0 \\ ^{-M_{JJ}} & I_{JJ} \end{bmatrix}. \]

\[ \det A(J) = \det A(J)^* = (-1)^p \det ^{M_{JJ}}. \]
3. LEMKE'S ALGORITHM

Lemke's Algorithm is an iterative method of processing¹ the linear complementarity problem \((q, M)\). Although the algorithm terminates in a finite number of steps, it may fail to process the linear complementarity problem for some classes of matrices \(M\). However, Eaves [3] has shown that Lemke's Algorithm will process the linear complementarity problem for a large class of matrices.

3.1. Preliminaries; Definitions and Notation

Consider the linear complementarity problem

\[(q, M): \text{Find } w, z \in \mathbb{R}^n\]

Subject to \(w = Mz + q\)

\[w \geq 0, z \geq 0\]

\[w^T z = 0.\]

¹/ Processing the linear complementarity problem means finding a complementary solution or showing that none exists.
If $q \geq 0$, then $w = q$, $z = 0$ is a complementary solution of $(q, M)$. Assume $q \neq 0$. Introduce an artificial variable $z_0$ and define the artificial problem

$$(q, M)^*:\quad \text{Find } w, z \in \mathbb{R}^n, z_0 \in \mathbb{R}$$

Subject to $w = Mz + q + z_0 e_n$

$$w \geq 0, \quad z \geq 0, \quad z_0 > 0$$

$$w^T z = 0$$

where $e_n$ is the n-vector $[1, 1, \ldots, 1]^T$.

Remark: $w = w_0$, $z = z_0$, $z_0 = 0$ constitute a solution of $(q, M)^*$ iff $w = w_0$, $z = z_0$ constitute a complementary solution of $(q, M)$.

The artificial problem can be rewritten as follows:

$$(q, M)^*:\quad \text{Find } w, z \in \mathbb{R}^n, z_0 \in \mathbb{R}$$

Subject to $[e_n^T \quad M \quad -I] z_0 = -q$ (21)

$$\begin{bmatrix} z_0 \\ z \\ w \end{bmatrix} \geq 0$$ (22)

$$w^T z = 0$$ (23)
3.1.1 Definition. A vector $[z_0, z_1, w]^T$ that satisfies (21) and (22) is called a feasible solution of $(q, M)^*$. 

Remark: Note that the matrix $[e_n^T, M; -I]$ is of rank $n$; hence, a basic feasible solution of $(q, M)^*$ has $n$ basic variables.

3.1.2 Definition. A basic feasible solution of $(q, M)^*$ is called complementary iff $z_0$ is nonbasic and exactly one variable in each pair $\{w_i, z_i\}$, $i = 1, 2, \ldots, n$, is basic.

3.1.3 Definition. A basic feasible solution of $(q, M)^*$ is called almost-complementary iff $z_0$ is basic and except for one index, say $i=k$, exactly one variable in each pair $\{w_i, z_i\}$, $i = 1, 2, \ldots, n$, if $k$, is basic. The pair $\{w_k, z_k\}$ is called a nonbasic pair.

Remarks: (1) Both complementary and almost-complementary basic feasible solutions satisfy the constraints of $(q, M)^*$.

(2) A complementary basic feasible solution of $(q, M)^*$ is a complementary solution of $(q, M)$.

3.1.4 Theorem. Let $q \in \mathbb{R}^n$ such that $q \neq 0$. Define the index $k$ by

$$q_k = \min_{1 \leq i \leq n} q_i.$$
Then

\[ z_0 = -q_k \]
\[ z = 0 \]
\[ w_i = q_i - q_k \quad (i=1, 2, \ldots, n) \]

constitute an almost complementary basic feasible solution with \( \{w_k, z_k\} \) as the nonbasic pair.

Proof: Note that

\[ (i) \quad z_0 e_n + Mz - w = -q_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + M \cdot 0 - \begin{bmatrix} q_1 - q_k \\ q_2 - q_k \\ \vdots \\ q_n - q_k \end{bmatrix} = -q. \]

\[ (ii) \quad z_0 = -q_k \geq 0 \]
\[ z = 0 \]
\[ w_i = q_i - q_k \geq 0, \quad i = 1, \ldots, n. \]

We show that the variables \( z_0, w_i \) (i\(\neq k)\), are basic. This is equivalent to showing that the column vectors associated
with these variables are linearly independent. The variables with their corresponding vectors are given below:

\[
\begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1
\end{bmatrix}, \begin{bmatrix}
  1 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}, \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  1
\end{bmatrix} \text{ kth coordinate}
\]

The linear independence of the vectors associated with \( z_0, w_j, j \neq k \), is now easy to show.

3.1.5. Definition. A basic feasible solution is said to be nondegenerate if all the basic variables are positive. The artificial problem \((q, M)^*\) is said to be nondegenerate if all the basic feasible solutions are nondegenerate.

3.2 Lemke's Algorithm

Assumptions: (1) \( q \neq 0 \) (2) \((q, M)^*\) is nondegenerate.

Notation: At the \( r \)th iteration, let the column associated with \( z_0 \) be denoted by \( Z_{r_0} \), the column associated with \( z_j (j=1, 2, \ldots, n) \) by \( Z_{r_j} \), the column associated with \( w_j (j=1, 2 \ldots, n) \) by \( W_{r_j} \), and the right hand side (RHS) by \( q^r \).
Step 0.1. **Initialization.** Set up the initial tableau:

<table>
<thead>
<tr>
<th></th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>...</th>
<th>$z_n$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>...</th>
<th>$w_n$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$M_{11}$</td>
<td>$M_{12}$</td>
<td>$M_{13}$</td>
<td>...</td>
<td>$M_{1n}$</td>
<td>-1</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>-q</td>
</tr>
<tr>
<td>1</td>
<td>$M_{21}$</td>
<td>$M_{22}$</td>
<td>$M_{23}$</td>
<td>...</td>
<td>$M_{2n}$</td>
<td>0</td>
<td>-1</td>
<td>...</td>
<td>0</td>
<td>-q_2</td>
</tr>
<tr>
<td>...</td>
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</tr>
<tr>
<td>1</td>
<td>$M_{n1}$</td>
<td>$M_{n2}$</td>
<td>$M_{n3}$</td>
<td>...</td>
<td>$M_{nn}$</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>-1</td>
<td>-q_n</td>
</tr>
</tbody>
</table>

0.2. Determine $k$ by

$$q_k = \min_{1 \leq i \leq n} \{ q_i \}.$$  

0.3. Pivot on $Z_{k0}$.

0.4. Multiply row $i$, $i \neq k$, by -1.

**Remark:** This initial step sets the tableau in feasible canonical form with respect to the first almost-complementary basic sequence

$$\langle z_0, w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n \rangle.$$  

0.5. Set $r=1$, $j=k$, $Z^r_{j} = A^r_{ij}$. Let $z_j$ be the incoming variable.
Step 1. Determine the index $t$ according to the equation

$$\frac{q_t^r}{A_{tj}^r} = \min \left\{ \frac{q_i^r}{A_{ij}^r} \left| A_{ij}^r > 0, 1 \leq i \leq n \right. \right\}$$

If $A_{ij}^r \leq 0$ for all $i = 1, 2, \ldots, n$, stop.
The incoming variable can be increased indefinitely without driving any of the basic variables towards zero. Otherwise, go to step 2.

Step 2. Pivot on $A_{tj}^r$.

2.0. If $z_o$ drops from the basis, stop. The new basic feasible solution is complementary.

2.1. If $z_j$, $j \neq 0$ drops from the basis, set $w_{j}^{r+1} = A_{r}^{+1}$, let $w_j$ be the incoming variable and go to step 1.

2.2. If $w_j$ drops from the basis, set $z_{j}^{r+1} = A_{r}^{+1}$, let $z_j$ be the incoming variable and go to step 1.

Remarks: (1) The ratio test in Step 1 is similar to the ratio test in the Simplex Algorithm in linear programming. It ensures that the basic solution at each iteration remains feasible.
(2) The stopping rule in Step 1 is also similar to the unboundedness stopping rule in the Simplex Algorithm. In this case, Lemke's Algorithm terminates in a ray. It will be shown that this stopping rule in Lemke's Algorithm fails to give any definite conclusion on the existence of a complementary solution for an arbitrary matrix M. However, for some classes of matrices (Eaves [3]), this stopping rule indicates that the linear complementarity problem (q,M) has no complementary solution.

3.3 Finiteness of Lemke's Algorithm

This section will show that Lemke's Algorithm terminates in a finite number of iterations.

3.3.1. Definition. If \( P_{r+1} \) is a basic feasible solution obtained from a basic feasible solution \( P_r \) by pivoting in Step 2 of Lemke's Algorithm, then \( P_r \) and \( P_{r+1} \) are said to be neighbors. The point \( P_r \) is called the predecessor of \( P_{r+1} \) and \( P_{r+1} \) is called a successor of \( P_r \).

3.3.2. Definition. The sequence of almost-complementary basic feasible solutions generated by Lemke's Algorithm is called an almost-complementary path.
3.3.3. Definition. Let \( P \) be an almost-complementary basic feasible solution with \( \{w_k, z_k\} \) as the nonbasic pair. If \( w_k \) or \( z_k \) can be increased indefinitely without driving any basic variable to zero, the ray emanating from \( P \) generated by increasing \( w_k \) or \( z_k \) indefinitely is called an almost-complementary ray.

3.3.4. Lemma. An almost-complementary basic feasible solution in an almost-complementary path can have at most two neighbors.

Proof: This follows from the fact that every almost-complementary basic feasible solution has exactly one nonbasic pair, say \( \{w_k, z_k\} \), and pivoting requires holding all nonbasic variables at value zero and increasing \( w_k \) or \( z_k \).

3.3.5. Lemma. The initial almost-complementary basic feasible solution cannot have a predecessor.

Proof: Let \( P_0 \) be the initial almost-complementary basic feasible solution with \( \{w_k, z_k\} \) as the nonbasic pair. Note that the successor of \( P_0 \) is obtained by pivoting \( z_k \) into the basis. Hence, if \( P_0 \) has a predecessor, then it can be obtained by pivoting \( w_k \) into the basis. We claim
that we cannot pivot $w_k$ into the basis, i.e., $w_k$ can be increased indefinitely without driving a basic variable to zero.

The relevant portion of the initial tableau follows:

\[
\begin{array}{cccccc}
& z_0 & \ldots & w_k & \ldots & \text{Right Hand Side} \\
1 & 0 & \ldots & & -q_1 & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
pivot \text{ element} & 1 & \ldots & 1 & \ldots & -q_k \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
1 & 0 & \ldots & & -q_n & \\
\end{array}
\]

After pivoting $z_0$ into the basis (with pivot element as shown above), the initial tableau in feasible canonical form with respect to the first almost-complementary basic sequence is:

\[
\begin{array}{cccccc}
& z_0 & \ldots & w_k & \ldots & \text{Right Hand Side} \\
0 & -1 & \ldots & & -q_1 + q_k & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
1 & \ldots & -1 & \ldots & -q_k \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & -1 & \ldots & & -q_n + q_k & \\
\end{array}
\]

Since the column associated with $w_k$ is negative, $w_k$ can be increased indefinitely without driving any basic variable to zero.

3.3.6. Theorem. Along an almost-complementary path, an almost complementary basic feasible solution cannot recur.
Proof: Let $P_0, P_1, \ldots$ be an almost complementary path and let $P_j$ be the first almost-complementary basic feasible solution that recurs. Let $P_k$ be $P_j$'s predecessor when it recurs. First, we note that $P_j \neq P_0$ since $P_0$ does not have a predecessor.

By the nondegeneracy assumption, the extreme points of the feasible region of $(q,M)^*$ are in one-to-one correspondence with the basic feasible solutions. Hence, $P_{j-1}, P_{j+1}$, and $P_k$ are distinct basic feasible solutions that are neighbors of $P_j$. This is impossible since $P_j$ can have at most two neighbors.
3.3.7. **Theorem.** Lemke's Algorithm terminates in a finite number of steps.

**Proof:** Starting from an almost-complementary basic feasible solution, Lemke's Algorithm generates an almost-complementary path and stops when (a) it generates an almost-complementary ray or (b) it obtains a complementary basic feasible solution. Noting that no almost-complementary basic feasible solution can recur and the number of basic feasible solutions is finite, the algorithm must terminate in a finite number of steps.

**Example 1.** This example shows how Lemke's Algorithm is used to solve a linear programming problem. It also shows termination of Lemke's Algorithm in a complementary solution.

**Primal Problem**

Min $144x_1 + 108x_2$

s.t. $4x_1 + x_2 \geq 4$

$3x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

**Dual Problem**

Max $4y_1 + 6y_2$

s.t. $4y_1 + 3y_2 \leq 144$

$y_1 + 3y_2 \leq 108$

$y_1, y_2 \geq 0$
Let $v = [v_1, v_2]^t$ be the vector of surplus variables for the primal problem and let $u = [u_1, u_2]^t$ be the vector of slack variables for the dual problem. The associated linear complementarity problem is that of finding $w, z \in \mathbb{R}^4$ such that $w = Mz + q$, $w \geq 0$, $z \geq 0$ and $w^Tz = 0$ where

$$
\begin{align*}
M &= \begin{bmatrix}
0 & 0 & -4 & -3 \\
0 & 0 & -1 & -3 \\
4 & 1 & 0 & 0 \\
3 & 3 & 0 & 0
\end{bmatrix} \\
q &= \begin{bmatrix}
144 \\
108 \\
-4 \\
-6
\end{bmatrix}
\end{align*}
$$

Initial Tableau

<table>
<thead>
<tr>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-144</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-108</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>6</td>
</tr>
</tbody>
</table>

Min $q_i = q_4 = -6$  $\forall i \leq 4$

Initial nonbasic pair = $\{z_4, w_4\}$

Initial almost complementary basic sequence: $\langle z_0, w_1, w_2, w_3 \rangle$
Introduce $z_3$ (the complement of $w_3$) into the basis; $w_1$ drops from the basis.

Introduce $z_1$ (the complement of $w_1$) into the basis; $z_0$ drops from the basis.
Since $z_0$ dropped from the basis, the algorithm stops and the following complementary solution is obtained:

\[
\begin{align*}
    w_1 &= 0 & z_1 &= \frac{2}{3} \\
    w_2 &= 0 & z_2 &= \frac{4}{3} \\
    w_3 &= 0 & z_3 &= 12 \\
    w_4 &= 0 & z_4 &= 32
\end{align*}
\]

In terms of the given linear programming problems,

\[
\begin{align*}
    u_1 &= w_1 = 0 & x_1 &= z_1 = \frac{2}{3} \\
    u_2 &= w_2 = 0 & x_2 &= z_2 = \frac{4}{3} \\
    v_1 &= w_3 = 0 & y_1 &= z_3 = 12 \\
    v_2 &= w_4 = 0 & y_2 &= z_4 = 32
\end{align*}
\]

The optimal solution of the primal problem is

\[
\begin{align*}
    x_1 &= \frac{2}{3} \\
    x_2 &= \frac{4}{3}
\end{align*}
\]

The optimal solution of the dual problem is

\[
\begin{align*}
    y_1 &= 12 \\
    y_2 &= 32
\end{align*}
\]
Example 2. This example shows Lemke's Algorithm terminating in an almost-complementary ray and the linear complementarity problem has no complementary solution.

\[
M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}
\]

Figure 4.

Note: The nondegenerate complementary cones are indicated by the curved arrows. The degenerate complementary cone \(\text{Pos}[-M_1; -M_2]\) is indicated by the bold line.
Since \( q \) does not belong to any complementary cone, the linear complementarity problem \((q, M)\) has no complementary solution.

\[
\begin{align*}
\text{Min } & \quad q_1 = q_2 = -2 \\
1 \leq i \leq 2
\end{align*}
\]

Initial nonbasic pair: \(\{w_2, z_2\}\)

Initial almost-complementary basic sequence: \(\langle z_0, w_1 \rangle\)

### Initial Tableau

<table>
<thead>
<tr>
<th></th>
<th>(z_0)</th>
<th>(z_1)</th>
<th>(z_2)</th>
<th>(w_1)</th>
<th>(w_2)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(-1m)</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

### Initial Tableau (Feasible Canonical Form)

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>(z_0)</th>
<th>(z_1)</th>
<th>(z_2)</th>
<th>(w_1)</th>
<th>(w_2)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w_1)</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(z_0)</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Introduce \(z_2\) (the complement of \(w_2\)) into the basis; \(w_1\) drops from the basis.
Introduction $z_1$ (the complement of $w_1$) into the basis; since the column under $z_1$ is nonpositive, the algorithm terminates in an almost-complementary ray.

**Remark:** The matrix $M$ in the preceding example is a positive semidefinite matrix. It has been shown (Eaves [3]) that if $M$ is positive semidefinite, then termination in an almost-complementary ray implies that the linear complementarity problem does not have a complementary solution. In fact, Eaves [3] has shown that this is true for a large class matrices that includes the positive semidefinite matrices.
Example 3. This example shows that Lemke's Algorithm can terminate in an almost-complementary ray even if the linear complementarity problem has a complementary solution. Let

\[ M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \]

Figure 5 illustrates the case of the matrix \( M \) and vector \( q \) given above. The point \( q \) is contained in the complementary cone \( \text{Pos}[-M_1, -M_2] \). Hence, \((q, M)\) has a complementary solution.
**Initial Tableau**

<table>
<thead>
<tr>
<th></th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Min $q_i = q_2 = -2$

$1 \leq i \leq 2$

Initial almost-complementary basic sequence: $\langle z_0, w_1 \rangle$

Initial nonbasic pair: $\{w_2, z_2\}$

**Initial Tableau (Feasible Canonical Form)**

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$z_0$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Introduce $z_2$ (the complement of $w_2$) into the basis.

Since the column under $z_2$ is nonpositive, stop. The variable $z_2$ can be increased indefinitely without driving any of the basic variables $z_0$ or $w_1$ to zero.

Hence, Lemke's algorithm terminates in an almost-complementary ray.

\[
\begin{align*}
\text{Min } f(x_1, x_2) &= x_1 + x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2 \\
\text{s.t. } x_1 + 2x_2 &\geq 2 \\
x_1, x_2 &\geq 0
\end{align*}
\]

This problem can be written as follows:

\[
\begin{align*}
\text{Min } [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} [x_1x_2] \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\text{s.t. } [1 \ 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\geq 2 \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\geq 0,
\end{align*}
\]

In the notation of Section 1.3,

\[
\begin{align*}
c &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
Q &= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \\
b &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
A &= \begin{bmatrix} 1 & 2 \end{bmatrix}
\end{align*}
\]
The objective function is convex since the matrix Q is positive definite. The associated linear complementarity problem \((q, M)\) has:

\[
q = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{bmatrix}
\]

\[
w = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \end{bmatrix}, \quad u = Qx - A^T y_1 + c \quad \text{and} \quad v_1 = Ax - b
\]

\[
z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} \quad \text{with} \quad y_1 = \text{Kuhn-Tucker multiplier}
\]

**Initial Tableau**

\[
\begin{array}{ccccccc|c}
\text{z}_0 & \text{z}_1 & \text{z}_2 & \text{z}_3 & \text{w}_1 & \text{w}_2 & \text{w}_3 & \text{RHS} \\
1 & 4 & 2 & -1 & -1 & 0 & 0 & -1 \\
1 & 2 & 4 & -2 & 0 & -1 & 0 & -1 \\
1 & 1 & 2 & 0 & 0 & 0 & -1 & 2 \\
\end{array}
\]

\[
\text{Min} \sum_{i=1}^{3} \text{q}_i = \text{q}_3 = -2
\]

Initial nonbasic pair: \(\{w_1, z_3\}\)

Initial almost-complementary basic sequence: \(z_0, w_1, w_2\).
### Initial Tableau (Feasible Canonical Form)

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td></td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$z_0$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Introduce $z_3$ (the complement of $w_3$) into the basis; $w_2$ drops.

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0</td>
<td>$\frac{5}{2}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$z_3$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$z_0$</td>
<td>1</td>
<td>1</td>
<td>$\boxed{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Introduce $z_2$ (the complement of $w_2$) into the basis; $z_0$ drops;

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>$z_0$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{5}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{5}{2}$</td>
</tr>
<tr>
<td>$z_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$z_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Since $z_0$ dropped from the basis, the algorithm stops at
a complementary solution given by:

- $w_1 = \frac{1}{2}$, $z_1 = 0$
- $w_2 = 0$, $z_2 = 1$
- $w_3 = 0$, $z_3 = \frac{5}{2}$

In terms of the given convex quadratic programming problem,
the optimal solution is given by

$$x_1 = 0, z_1 = 0$$
$$x_2 = 1, z_2 = 1$$

**optimal solution,** $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$f(x^*) = 3$$

**Figure 6**
3.4 Termination of Lemke's Algorithm in a Ray

Examples 2 and 3 show that termination in a ray does not lead to a definite conclusion on the existence of a complementary solution. However, for some classes of matrices $M$, termination in a ray implies that the linear complementarity problem has no complementary solution. Lemke proved this result for the class of copositive plus matrices which include the positive semidefinite matrices. We note that the matrices in the linear complementarity problems associated with the linear programming problem and the convex quadratic programming problem are of the form

$$
\begin{bmatrix}
0 & -A^t \\
A & 0 \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
Q & -A^t \\
A & 0 \\
\end{bmatrix}
$$

which are easily shown to be positive semidefinite. It follows that Lemke's Algorithm can solve linear programming and convex quadratic programming problems. This section shows that positive semidefinite matrices are copositive plus.

3.4.1 Definition. A square matrix $M$ is copositive plus iff

(i) $z^tMz \geq 0$ for all $z \geq 0$

(ii) $(M + M^t)z = 0$ if $z^tMz = 0$ and $z \geq 0$. 
3.4.2 **Lemma.** If $M$ is a real symmetric matrix, then there exists a nonsingular matrix $U$ such that $U^tMU$ is a diagonal matrix whose diagonal entries are characteristic roots of $M$.

Proof: See Höhn [4].

3.4.3 **Lemma.** If $M$ is a real symmetric positive semidefinite matrix and $z^tMz = 0$, then $Mz = 0$.

Proof: Let $M$ be an $n \times n$ real symmetric positive semidefinite matrix. Let $U$ be the nonsingular matrix in Lemma 3.4.2 and let

$$U^tMU = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix} = \Lambda$$

where $\lambda_i \geq 0$ ($i = 1, 2, \ldots, n$) are the characteristic roots of $M$. Define

$$Z = U^{-1}z.$$ 

Then

$$z = UZ$$

By assumption, $z^tMz = 0$, i.e.,

$$z^tU^tMUZ = 0$$

$$z^tAz = 0$$

where

$$\sum_{i=1}^{n} \lambda_i z_i^2 = 0.$$
Since \( \lambda_i z_i^2 \geq 0 \) for \( i = 1, 2, \ldots, n \), then

\[
\lambda_i z_i^2 = 0, \quad i = 1, 2, \ldots, n;
\]

hence, \( \lambda_i z_i = 0, \quad i = 1, 2, \ldots, n \).

Now, \( U^T M U \bar{z} = \Lambda \bar{z} = \begin{bmatrix} \lambda_1 \bar{z}_1 \\ \lambda_2 \bar{z}_2 \\ \vdots \\ \lambda_n \bar{z}_n \end{bmatrix} = 0 \)

Since \( U^T \) is nonsingular, then

\( M \bar{z} = 0 \)

or

\( Mz = 0 \).

3.4.4 Theorem. Let \( M \) be a real positive semidefinite matrix.

If \( z > 0 \) and \( z^T M z = 0 \), then \( (M + M^T)z = 0 \).

Proof: If \( M \) is positive semidefinite then \( M + M^T \) is symmetric positive semidefinite and

\[ \frac{1}{2} z^T (M + M^T)z = z^T Mz. \]

If \( z^T Mz = 0 \), then \( z^T (M + M^T)z = 0 \).

By Lemma 3.4.3

\[ (M + M^T)z = 0. \square \]

3.4.5 Corollary. A positive semidefinite matrix is copositive plus.

Proof: Follows from the definition of a positive semidefinite matrix and Theorem 3.4.4. \( \square \)
REFERENCES


