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by

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Abstract

This study presents a more general collusive mechanism that is sustainable in an oligopolistic repeated game. In this setup, firms can obtain average payoffs beyond the cooperative profits while at the same time improve consumer welfare through a lower market price offer. In particular, we introduce here the notion of intertemporal collusive trade where each oligopolist, apart from regularly producing the normal cooperative output, is also allowed in a systematic way to "deviate" and earn higher than the rest at some stages of the game. This admits subgame-perfection and is shown under some conditions to be Pareto-superior to the typical cooperative outcome.

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The irony of commitment is that it's deeply liberating ...
- Anne Morriss

1. Introduction

The basic idea of intertemporal collusion is that firms are able to sustain themselves over time in being faithful to a contracted level of production. Oligopolists, in general, are inclined to collaborate with one another by targeting a certain level of production that will yield the highest possible profit for each one. The main difficulty however is that any firm will always face a temptation to produce more than what was agreed upon so as to extract even higher profits, thereby making any form of pre-game commitment unsustainable. As a consequence, everyone acts strategically and so the market settles at a Cournot-Nash equilibrium where profits are lower than when collusion had been made possible.

The literature on repeated games however offers a well known solution to this problem by asserting that stable collusion is attainable whenever firms interact over a long period of time. They can employ the so called *trigger strategies* where everyone starts by producing collusive outputs and continues doing so for as long as everyone remains loyal to the contract. The moment any one of them cheats, everyone knows that everyone will revert to the Cournot-Nash production as a form of punishment (Friedman, 1971). This imposition of credible threat is enough to discourage any form of deviation at any point in time and is the heart of the subgame perfection principle (Selten, 1975).

While the notion of subgame perfect equilibrium features a Cournot-Nash punishment that is sufficient to induce fidelity to the contract, it still carries a mild deterrence power that could give way to some *renegotiation* once a deviation has occurred. In other words, after a deviation firms may say "let's forget about the past and bring back the good ol' days" by not punishing ourselves forever. Several studies have tackled this issue (*e.g.* Abreu (1986), Fudenberg & Maskin (1986), and Farrell & Maskin (1989)) and showed that by administering a shorter yet more intense credible punishment, greater possibilities for stronger cooperation can take place.

Our goal in this paper is to explore the extent of feasible stable cooperation in an oligopolistic repeated game in a manner quite different from what has been studied so far. In characterizing cooperative outcomes, we dwell more here on the design of collusive contract rather than on penal structures. Rather than highlighting the need for a severe punishment program that will deter any possible deviation, we turn towards an incentive mechanism that will elicit greater cooperative potentials. As it is not in our interest to impose the "best" punishment scheme, the use of Cournot-Nash reversion is sufficient in our study in order to highlight more the construction of sustainable contracts, making also our presentation simple and concise.[2]

Specifically, we introduce the notion of intertemporal collusive trade where firms, while they continue to produce collusive outputs, are legally allowed to "deviate" at some prescribed stages so as to earn more profits. Our approach here in finding greater possibilities for stable cooperation is through a *trading of payoffs over time* that encourages firms to be faithful to the contract rather than on the severity of punishment that discourages any form of deviation. In this system, firms may be getting less during the regular stages by allowing someone else to "deviate", but the thought of also obtaining privileged payoffs in the future (one after another) is a motivation for everyone to stick to the program. It is shown that when this intertemporal trading is performed in a recurring fashion, similar to the method of Fudenberg and Maskin (1991), subgame-perfect equilibrium can be achieved. Equally important is the result that in this setup, consumer welfare is upgraded through a lower market price level, as induced by higher aggregate production.

This paper also contributes to the literature on market structure by showing that every firm, under this trading system, can obtain profits higher than in the normal intertemporal collusive equilibrium. This seems to stand in contrast with the common notion that a shared monopoly profit is always the profit-maximizing scheme available for oligopolists. While this remains true when firms have uniform discount factor, it no longer holds under differentiated discount factors since the possibilities for stable intertemporal trading of payoffs become richer. For example, by constructing a program where those who have less capacity to wait receive their privileged payoffs ahead of those who are more willing to wait, everyone's average payoff can be made higher than the typical collusive income. This approach is an application to the main message of Lehrer and Pauzner (1999), which establishes that the difference

[2] Abreu's optimal punishment scheme is nonetheless presented in the Appendix A for completeness, incorporating it with the contract mechanism that is introduced in this paper.

in discount factors between two players can create *new cooperative possibilities* that broadens the set of sustainable repeated-game payoffs.

The rest of the paper is organized as follows. Section 2 reviews the Cournot-Nash benchmark model and introduces some notations. Section 3 presents the intertemporal collusive trade model and its conditions for stability while Section 4 provides results on comparative statics. Section 5 extends the setup to some generalizations and establishes its payoff-dominance over the typical collusive outcome. Section 6 concludes.

2. Cournot-Nash Benchmark

Consider a set of n oligopolists who simultaneously choose to produce a certain quantity of homogenous good. Denote this quantity produced as q_i by firm i and write the aggregate demand function as $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, in which every price $p \in \mathbb{R}_+$ induces a total demand $D(p)$. We assume that the inverse of this function, denoted by $p(\cdot)$, is strictly monotonic and continuous and that the constant marginal cost is the same for all firms. In a usual fashion, we then express the profit function as:

$$\pi_i (q_1, q_2, ..., q_n) = p\left(\sum_{j=1}^{n} q_j\right) q_i - cq_i \tag{1}$$

Suppose there exists a unique and symmetric Cournot-Nash equilibrium wherein every firm produces q^c units of output with a corresponding profit of π^c, *i.e.*

$$\pi^c = \pi_i \left(q_i^c, q_{-i}^c\right) \geq \pi_i \left(q_i', q_{-i}^c\right), \tag{2}$$

for all $q_i' \geq 0$ and where $q_i' \neq q_i^c$ and $q_{-i}^c = \left(q_1^c, ..., q_{i-1}^c, q_{i+i}^c, ..., q_n^c\right)$. [3]

Producing q^c therefore is seen as the best-response strategy of every firm in a noncooperative single-shot game. On the contrary, if firms were to collude, they will maximize their individual profit π^* by aiming at a monopoly production nq^*, where the uniform production q^* is defined as:

[3] To simplify our notation, the unsubscripted symbols of q and π shall refer to q_i and π_i while their corresponding bold symbols shall denote a vector across firms, *i.e.* $\mathbf{q} = (q_1, q_2, q_3, ..., q_n)$.

$$q^* = \arg\max_{q \geq 0} \pi_i(\mathbf{q}) \tag{3}$$

In a traditional manner, we assume that q^* for all firms is unique such that $\pi(\mathbf{q})$ is strictly lower when q is either greater or lower than q^*. The catch is that the commitment to produce q^* for each firm is not sustainable since there is always an incentive for anyone to cheat and produce more than what was agreed upon. Let us denote $\bar{\pi}$ as the profit obtained from unilaterally deviating from the contract by producing $\bar{q}$ units of output. More formally, we define $\bar{q}$ as:

$$\bar{q} = \arg\max_{q_i \geq 0} \pi_i\left(q_i, q_{-i}^*\right), \tag{4}$$

$$\text{where } q_{-i}^* = \left(q_1^*, ..., q_{i-1}^*, q_{i+i}^*, ..., q_n^*\right).$$

Now, consider an infinitely repeated game $\Gamma^{\infty}(\delta, s_i^t(h^t), \pi)$, where $\delta \in (0,1)$ is the common discount factor, $s_i^t(h^t)$ is the pure strategy of firm i at time t given the history of past actions $h^t = (\mathbf{q}^1, \mathbf{q}^2, ..., \mathbf{q}^{t-1})$, where $h^1 = \varnothing$, and π as discussed is the continuous payoff function. Note that this history is public and shared by all players at every stage of the game. In the typical trigger strategy, each firm's production strategy is defined as follows:

$$s_i^t(h^t) = \begin{cases} q^*, & \text{if at } t=1 \text{ and if at } t \geq 2,\ s_i^{t-1}(h^{t-1}) = q^* \text{ for all } i \\ q^c, & \text{otherwise} \end{cases} \tag{5}$$

The stream of quantity-produced across firms is therefore depicted by $\{\mathbf{s}^t(h^t)\}_{t=1}^{\infty}$ with its associated average discounted payoff over time for each firm as:

$$\Pi_i(\mathbf{s}^t(h^t)) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_i(\mathbf{s}^t(h^t)) \text{ [4]} \tag{6}$$

This strategy in an infinitely-repeated game can now sustain the collusive production of q^* given the threat of reverting to the Cournot-Nash production of q^c once any of the player deviates. Sustainability is made possible since the Cournot-Nash

[4] The computation for discounted income makes use of the following formula: $1 + \delta + \delta^2 + ... + \delta^{n-1} = \frac{1-\delta^n}{1-\delta}$.

punishment scheme, with δ sufficiently high, is a subgame perfect equilibrium that enforces credibility of threat.

At this juncture, one may wonder whether the goal of consistently producing q^* to earn an average discounted profit of Π^* is the maximum earnings that a firm could get in any repeated-game setup. In what follows, we show that oligopolists can still improve their earnings beyond q^* while at the same time provide greater consumer welfare in terms of lower price offer.

3. The Setup: Intertemporal Collusive Trade

We introduce here the notion of intertemporal collusive trade which maintains a production of q^* units of goods for each of the $n-1$ firms while allowing a single firm to produce its best-response production of $\bar{q}$. The main idea of this setup is to allow one firm at a time to "deviate" from producing q^* in order for that firm to earn higher than the rest on that particular stage. We show then that this system is sustainable (*i.e.* subgame perfect) when performed repeatedly over infinite horizon. Formally, we define this strategy as follows:

Definition 1. An *intertemporal collusive trade* (ICT) strategy is a strategy profile $\{\mathbf{s}^t(h^t)\}_{t=1}^{\infty}$ where

(i) each firm $i \in N = \{1, 2, 3, ..., n\}$ precommits itself at the start of the game to a production profile $q_i(t)$ defined as follows:

$$q_i(t) = \begin{cases} \bar{q}\ , & \text{for all } t = i + nz, \text{ where } z \in \{0, 1, 2, ...\} \\ q^*, & \text{for all } t \neq i + nz, \text{ where } z \in \{0, 1, 2, ...\} \end{cases}$$

(ii) and if at all $t' < t$, where $t \geq 2$, $s_i^{t'}(h^{t'}) = q_i(t')$ for all $i \in N$, then $s_i^t(h^t) = q_i(t)$. Otherwise, $s_i^t(h^t) = q_c$ for all $i \in N$.

Call the stage *privilege stage* when firm i is expected to produce its best-response $\bar{q}$ and call it *regular stage* when i is supposed to produce q^*.

While the second part of the definition is the familiar Cournot-Nash punishment imposed to any unilateral deviation from the strategy, the first part requires some explanation. In this pre-game setup, each player i is termed as the i^{th} player in the order of succession, such that $i = 1$ being the first player and $i = n$ being the last in a cycle that is infinitely repeated. Each firm i is scheduled to have its highest (privileged) production on the i^{th} stage (*i.e.* $t = i$) and on the succeeding rounds of n-stage interval. During regular times, he produces q^* along with other $n-2$ firms. Thus, at every stage there is always one who produces $\bar{q}$ while there are $n-1$ firms who produce q^*. While we denote as before the stage-profit of that solitary firm as $\bar{\pi}$, we have $\pi' = \pi_{j \in N \setminus \{i\}}(\bar{q}_i, q^*_{-i})$ as the profit of those who produce q^* at every stage. It is easy to see that by allowing one firm to produce $\bar{q}$, total production increases and so the others no longer reach the normal collusive profit π^* but settle at a lower payoff π', thus we have $\bar{\pi} > \pi^* > \pi'$. Finally, we assume here a perfect monitoring environment where any deviation from the strategy can be observed by any firm and at any time.

The first part of the definition is an open-loop strategy that maps out the moves of every firm in a calendar time (see Fudenberg and Levine, 1988). However, we are interested here in a subgame-perfect (closed-loop) equilibrium that considers the reaction of players to any possible deviation at any stage of the game. To argue therefore that the ICT strategy is subgame-perfect that yields superior profit to Cournot-Nash outcome, we need to show that its average discounted profit is higher than that obtained in a Cournot-Nash outcome and that there is no incentive for any player to deviate at any subgame of Γ^∞. We present these conditions formally through the following definition.

Definition 2.

(i) The *individual-rationality condition* (IRC) is satisfied if $\Pi_i(\mathbf{q}(t)) > \Pi^c(\mathbf{q}^c)$, for any $\mathbf{q} \geq (0, 0, ..., 0)$ and where $\Pi^c(\mathbf{q}^c) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi^c = \pi^c$.

(ii) The *incentive-compatibility condition* (ICC) is satisfied if

$\Pi_i(\mathbf{s}^t(h^t)) > \Pi_i\left(\mathbf{s}^t(h^t) \setminus \left[s_i^d(h^d), s_{-i}^t(h^t)\right]\right)$, where the right-hand side is the average entire-game payoff to i when i deviates from the strategy at time $d \in \{1, 2, ...\}$.

Remark. Notice that instead of the normal use of continuation payoffs, the ICC is depicted using the entire-game payoffs of those profiles which deviate at time d. While these two methods are equivalent, we use the latter which is computationally more convenient when dealing with ICT strategies.

It is important to highlight here that whenever the ICC holds, it is a sufficient claim that any deviation at any stage of the game will not be profitable, including those during the punishment regime. This is mainly because the Cournot-Nash penalty is always an equilibrium. Thus, the fact that any deviation during the punishment phase will not make anyone better off simplifies our study into just ensuring that the entire-game payoff derived from fulfilling any pre-game commitment is at least as much as the payoff of any profile that contains a deviation at any time with its subsequent punishment[5]. We express this formally as follows:

$$\Pi_i(\mathbf{q}(t)) \geq \Pi_i \left(\mathbf{q}(t) \setminus \left[\left(q^d(d), q_{-i}(d)\right), (q^c)_{d+1}^{\infty}\right]\right)$$

where again the right-hand side is the entire-game payoff to i given the production path that deviates from the commitment $\mathbf{q}(t)$ at time $d \in \{1, 2, ...\}$ followed by a punishment path from $d+1$ onwards. This now leads us to the following definition.

Definition 3. A strategy is sustainable in a subgame-perfect equilibrium if both the IRC and ICC are satisfied for every firm i.

Since our equilibrium concept requires the fulfillment of the IRC and ICC in each and every firm, the first two lemmas shall provide a characterization for the admissibility of these two conditions to all firms.

Lemma 1.

(i) If the IRC holds for the last firm n, then it also holds for all the preceding $(n-1)$ firms.

(ii) If $\pi' \geq \pi^c$, then the IRC is always satisfied for all i .

Proof:

(i) When there is no deviation from the pre-game commitment, the entire-

[5]More formally, we say that under a punishment regime, $\Pi_i \left(\mathbf{q}(t) \setminus \left(\left(q^d(d), q_{-i}(d)\right), (q^c)_{d+1}^{\infty}\right)\right) \geq \Pi_i \left(\left[\mathbf{q}(t) \setminus \left(\left(q^d(d), q_{-i}(d)\right), (q^c)_{d+1}^{\infty}\right)\right] \setminus \left(\left(q_i'(e), q_{-i}^c(e)\right), (q^c)_{e+1}^{\infty}\right)\right)$ remains always true, where the right hand side is the entire-game payoff derived from a production profile that deviates from the punishment path at time e, where $e > d$. Thus, one needs only to ensure that $\Pi_i(\mathbf{q}(t)) \geq \Pi_i \left(\mathbf{q}(t) \setminus \left(\left(q^d(d), q_{-i}(d)\right), (q^c)_{d+1}^{\infty}\right)\right)$ as mentioned in the text.

game average discounted earnings of the i^{th} player is given by:

$$\begin{aligned}\Pi_i &= (1-\delta)\left(\frac{\pi'(1-\delta^{i-1})}{(1-\delta)} + \sum_{T=0}^{\infty}\delta^{nT}\left(\bar{\pi}\delta^{i-1} + \sum_{t=i+1}^{i+n-1}\pi'\delta^{t-1}\right)\right), \\ &\quad \text{where } T = \{0,1,...\} \\ &= \pi'(1-\delta^{i-1}) + \frac{\bar{\pi}\delta^{i-1}(1-\delta) + \pi'\delta^{i}(1-\delta^{n-1})}{(1-\delta^{n})} \\ &= \pi' + \frac{(\bar{\pi}-\pi')(1-\delta)\delta^{i-1}}{1-\delta^{n}} \qquad (7)\end{aligned}$$

In (7), we see that Π_i monotonically decreases in i for $\delta \in (0,1)$; clearly the n^{th} player will earn the least profit. Thus, if $\Pi_n > \Pi^c$, then $\Pi_1, \Pi_2, ..., \Pi_{n-1} > \Pi^c$.

(ii) From (i), it is sufficient to check that $\Pi_n > \Pi^c$ in ensuring that all firms pass the IRC. This implies that $\pi' + \frac{(\bar{\pi}-\pi')(1-\delta)\delta^{n-1}}{1-\delta^{n}} > \pi^c$. Since $\bar{\pi} - \pi' > 0$, we obtain:

$$\frac{(1-\delta)\delta^{n-1}}{1-\delta^{n}} > \frac{\pi^c - \pi'}{\bar{\pi} - \pi'} \qquad (8)$$

Clearly, the left-hand side of the inequality is always positive for all $\delta \in (0,1)$ and $n < \infty$. Thus, (8) always holds whenever $\pi^c \leq \pi'$. *q.e.d.*

While the above lemma provides hint as to when IRC is satisfied by all players under an ICT strategy, the following one prepares the ground for the admissibility of ICC to everyone. To do this, one has to show that no firm would deviate at any point of the game since its overall average income when deviating at any stage is always less than what it obtains from simply sticking to the plan. Let us denote q^d, which induces an earning of π^d, as the deviatory production during stages when a firm is supposed to produce q^*, such that:

$$q^d = \arg\max_{q_i \geq 0} \pi_i(q_i(t), \bar{q}, (n-2)q^*), \text{ for all } t = s_i^{-1}(q^*) \qquad (9)$$

Note that q^d is only defined for those (regular) stages where i is expected to produce q^*. This is because during privilege stages, i receives already the highest possible profit and so there is no more incentive to deviate. To further simplify our investigation, note that there is no need also to determine every deviatory income one can obtain from each of those regular stages as long as one can pin point the stage that offers the highest incentive to firm i. If the entire-game payoff that consists deviation on that stage remains inferior to that of the ICT program, then that firm has passed the ICC.

In characterizing the admissibility of ICC to all firms, the main difficulty lies on the twofold asymmetry that exists across firms: one is on the average income obtained under no-deviation scenario and the other is on the highest entire-game incentive one could get from deviating. Interestingly, when these two are compared in each firm, the result is a uniform condition that governs all firms, as presented in the following lemma.

Lemma 2. The ICC is the same for every firm i and is characterized by the following inequality:

$$\frac{(\pi^d - \pi')}{(\bar{\pi} - \pi')} - \delta\frac{(\pi^d - \pi^c)}{(\bar{\pi} - \pi')} < \delta\frac{(1-\delta)}{(1-\delta^n)}.$$

For $\delta > \frac{\pi^d - \pi'}{\pi^d - \pi^c}$, where $\pi^d > \pi' > \pi^c$, the ICC is always satisfied.

Proof:

(Step 1) Any player i does not have any incentive to deviate on the $(i+nz)^{th}$ stages since the privileged production at $\bar{q}$ is the best-response to the q^* production of all the others (see (4)).

(Step 2) Now, consider the regular stages where firm i is supposed to produce q^*, particularly, only the stages from $i+1$ to $i+n-1$. To depict the entire-game payoff on any possible stage of deviation between stages $i+1$ and $i+n-1$, we define $\Pi_i^d(w)$ below where $w \in \{0, 1, ..., n-2\}$ represents the number of stages that q^* is produced (i.e. π' is earned) after the i^{th} stage and just before deviating to q^d. Note that when $w = n-1$, the deviation will occur on the $(i+n)^{th}$ stage, which we have already ruled out in Step 1.

$$\Pi_i^d(w) = (1-\delta)\left(\frac{\pi'(1-\delta^{i-1})}{(1-\delta)} + \bar{\pi}\delta^{i-1} + \sum_{t=i+1}^{i+w}\pi'\delta^{t-1} + \pi^d\delta^{i+w} + \sum_{t=i+w+2}^{\infty}\pi^c\delta^{t-1}\right) \quad (10)$$

In Lemma 8 (see Appendix B), it is shown that every m^{th} stage of every round/cycle in an infinitely repeated game generates the same condition for not deviating from the strategy. Thus, it is sufficient to study only the conditions for stages $i+1$ to $i+n-1$ since that would also depict the conditions of their corresponding stages in other rounds. Indeed, we have $\Pi_i^d(w) = \Pi_i^d(w+nz)$, for all $z \in \{1, 2, 3, ...\}$.

(Step 3) To pass the ICC, the average discounted profit of any player in a complete game without deviation must be higher than that of an event when a deviation occurs at any time of the infinely-repeated game, *i.e.* $\Pi_i > \Pi_i^d(w)$. Using (7) and (10), we restate this condition below:

$$\pi' + \frac{(\bar{\pi} - \pi')(1-\delta)\delta^{i-1}}{1-\delta^n} \quad > \quad \pi'(1-\delta^{i-1}) + \bar{\pi}\delta^{i-1}(1-\delta) + \pi'\delta^i(1-\delta^w) + \pi^d\delta^{i+w}(1-\delta) + \pi^c\delta^{i+w+1}$$

By simplifying, we obtain:

$$\Leftrightarrow \quad \frac{(\bar{\pi} - \pi')(1-\delta)\delta^{i-1}}{1-\delta^n} > (\bar{\pi} - \pi')\delta^{i-1} - (\bar{\pi} - \pi')\delta^i + (\pi^d - \pi')\delta^{i+w} - (\pi^d - \pi^c)\delta^{i+w+1}$$

$$\Leftrightarrow \quad (\bar{\pi} - \pi')(1-\delta)\delta^{i-1}\frac{\delta^n}{1-\delta^n} > \delta^{i+w}(\pi^d - \pi^c)\left(\frac{(\pi^d - \pi')}{(\pi^d - \pi^c)} - \delta\right)$$

$$\Leftrightarrow \quad \delta^{n-1}\frac{(\bar{\pi} - \pi')}{(\pi^d - \pi^c)}\frac{(1-\delta)}{(1-\delta^n)} > \delta^w\left(\frac{(\pi^d - \pi')}{(\pi^d - \pi^c)} - \delta\right), \quad \text{since } \pi^d > \pi^c$$

The last inequality does not contain the parameter i, which shows that the ICC is the same for all firms. Moreover, observe that for any $w \in \{0, 1, ..., n-2\}$, this condition is satisfied for $\delta > \frac{\pi^d - \pi'}{\pi^d - \pi^c}$. Finally, notice also that when $\delta > \frac{\pi^d - \pi'}{\pi^d - \pi^c}$, the right-hand side of inequality monotonically increases in w. Thus, every player i, if he were to deviate, will do it on the $(i+n-1)^{th}$ stage (or when $w = n-2$) since it is when it obtains the highest incentive. Thus, by setting $w = (n-2)$ on this inequality, we obtain the most stringent ICC and derive the assertion of the lemma. *q.e.d.*

We are now prepared to present our first proposition.

Proposition 1. Given that $\pi^d > \pi' > \pi^c$ and that δ is sufficiently high, an ICT strategy in an oligopolistic repeated game can be sustained in a subgame-perfect equilibrium.

Proof:

The proof follows directly from Lemma 1, Lemma 2, and Definition 3. *q.e.d.*

The result of the above proposition can be regarded weak in the sense that it rests mainly on the assumption that $\pi^d > \pi' > \pi^c$. In the next section, we explore under what conditions can this requirement be always made true.

4. Comparative Statics

In this section, we study how the possible stage-payoffs under an ICT strategy differ from one another with the changes in the number of firms in an oligopoly. We start by setting a benchmark inverse demand function $p = a - \sum_{i \in N} q_i$ that is strictly monotonic and continuous on an interval $[0, a]$ and which we assume to be above the marginal cost c in order that $\pi_i > 0$ for all choices of $q_i > 0$. Our payoff function therefore is defined as:

$$\pi_i = \left((a - c) - \textstyle\sum_{j \in N} q_j\right) q_i \tag{11}$$

Under a monopoly, the solitary firm will produce the quantity $q_m = \frac{a-c}{2}$ which yields the maximum profit of $\pi^m = \frac{(a-c)^2}{4}$. In a collusive oligopoly with n players, the maximum profit is attained by simply collaborating among themselves in maintaining the monopoly production, *i.e.* $q^* = \frac{a-c}{2n} = \frac{q_m}{n}$. Consequently, this leads to the per firm collusive profit of $\pi^* = \frac{(a-c)^2}{4n} = \frac{\pi^m}{n} = nq^{*2}$. However, as mentioned before, the collusive production is not sustainable in a simultaneous single-stage game and so everyone settles at the Cournot-Nash equilibrium with individual output of $q^c = \frac{a-c}{n+1} = \frac{2q_m}{(n+1)}$ and profit of $\pi^c = \frac{(a-c)^2}{(n+1)^2} = \frac{4q_m^2}{(n+1)^2}$. All assertions of equivalence in this paragraph are easily verifiable.

Under the ICT program, a firm has four possible choices of production: the privileged output $\bar{q}$, the Cournot output q^c, the collusive output q^*, and the deviatory output q^d. While firms do not have incentive to deviate when producing $\bar{q}$ being its best-response, there is always the possibility of producing the maximum deviation q^d

whenever one is supposed to produce q^*.To compute for q^d, we derive $\bar{q}$ first using (11) and (4):

$$\begin{aligned}\bar{\pi} &= ((a-c)-(\bar{q}+(n-1)q^*))\,\bar{q}\\ \frac{\partial\bar{\pi}}{\partial\bar{q}} &= (a-c)-2\bar{q}-(n-1)q^*=0\\ \bar{q} &= q^*\frac{(n+1)}{2}\end{aligned}$$

Then, by using (11) and (9), we have:

$$\begin{aligned}\pi^d &= \big((a-c)-(q^d+\bar{q}+(n-2)q^*)\big)q^d\\ \frac{\partial\pi^d}{\partial q^d} &= 2nq^*-2q^d-\frac{q^*(n+1)}{2}-(n-2)q^*=0\\ q^d &= q^*\frac{(n+3)}{4}\end{aligned}$$

From the derived q^d and $\bar{q}$, we obtain their respective profits:

$$\pi^d = q^{*2}\frac{(n+3)^2}{16}\ , \qquad \bar{\pi} = q^{*2}\frac{(n+1)^2}{4}$$

Proposition 2. Under the ICT strategy, the price offered to the market at every stage is lower than that of the normal collusion.

Proof:

Under the normal collusion, the price function is given by $p = a - nq^*$ while for ICT it is $p = a - ((n-1)q^* + \bar{q}) = a - \left(\frac{3n-1}{2}\right) q^*$. It is therefore clear that for $a > 0$ and $n > 1$, ICT offers lower price level than the normal collusion. *q.e.d.*

The table below summarizes the payoffs and outputs of each firm at every type of stage, along with the aggregate market output.

	Payoff of i	Output of i	Total Output
Privilege stage	$\bar{\pi} = q^{*2}\frac{(n+1)^2}{4}$	$\bar{q} = q^{*}\frac{(n+1)}{2}$	$Q_{(\bar{q},(n-1)q^*)} = q^{*}\frac{(3n-1)}{2}$
Regular stage	$\pi' = q^{*2}\frac{(n+1)}{2}$	q^*	$Q_{(\bar{q},(n-1)q^*)} = q^{*}\frac{(3n-1)}{2}$
Deviatory stage	$\pi^d = q^{*2}\frac{(n+3)^2}{16}$	$q^d = q^{*}\frac{(n+3)}{4}$	$Q_{(\bar{q},(n-2)q^*,q^d)} = q^{*}\frac{(7n-3)}{4}$
Punishment stage	$\pi^c = q^{*2}\frac{4n^2}{(n+1)^2}$	$q^c = q^{*}\frac{2n}{(n+1)}$	$Q_{(nq^c)} = q^{*}\frac{2n^2}{(n+1)}$

Table 1: Individual payoffs and outputs at different types of stages

In our next lemma, we show that the comparative structure of the different types of stage-payoff and stage-output becomes stable when the number of firms in an oligopoly increases. We present this result as follows:

Lemma 3.
(i) If $n \geq 5$, then $\bar{\pi} > \pi^d > \pi' > \pi^c$.
(ii) If $n \geq 4$, then $\bar{q} > q^d > q^c > q'$

Proof:

(i) a. Suppose $\pi^c \geq \pi'$. Then, $q^{*2}\frac{4n^2}{(n+1)^2} \geq q^{*2}\frac{(n+1)}{2} \Rightarrow 8n^2 \geq n^3 + 3n^2 + 3n + 1 \Rightarrow (n-1)(n^2 - 4n - 1) \leq 0$. Since $n > 1$, the admissible values of n are in the interval $(1, 2+\sqrt{5}]$. Thus, if $n > 2+\sqrt{5}$, then $\pi' > \pi^c$.

b. Suppose $\pi' \geq \pi^d$. Then, $q^{*2}\frac{(n+1)}{2} \geq q^{*2}\frac{(n+3)^2}{16} \Rightarrow 8n + 8 \geq n^2 + 6n + 9 \Rightarrow (n-1)^2 \leq 0$. The only solution here is $n = 1$. Thus, for $n > 1$, it must be that $\pi^d > \pi'$.

c. Suppose $\pi^d \geq \bar{\pi}$. Then, $q^{*2}\frac{(n+3)^2}{16} \geq q^{*2}\frac{(n+1)^2}{4} \Rightarrow n^2+6n+9 \geq 4(n^2+2n+1) \Rightarrow (3n+5)(n-1) \leq 0$. But $n \nleq 1$ and so $\bar{\pi} > \pi^d$.

From a-c and by the assumption of monotonic continuous payoff functions, we proved the first part of this lemma.

(ii) The proof is analogous to (i) and is therefore omitted. *q.e.d.*

The following table presents how the output and payoff of each firm vary with the total number of firms in an oligopoly. It also verifies the claim of Lemma 3.

Number	Payoff of i				Output of i			
of Firms	$\bar{\pi}$	π'	π^d	π^c	$\bar{q}$	q^*	q^d	q^c
2	$2.25q^{*2}$	$1.5q^{*2}$	$1.56q^{*2}$	$1.77q^{*2}$	$1.5q^*$	q^*	$1.25q^*$	$1.33q^*$
3	$4.00q^{*2}$	$2.0q^{*2}$	$2.25q^{*2}$	$2.25q^{*2}$	$2.0q^*$	q^*	$1.50q^*$	$1.50q^*$
4	$6.25q^{*2}$	$2.5q^{*2}$	$3.06q^{*2}$	$2.56q^{*2}$	$2.5q^*$	q^*	$1.75q^*$	$1.67q^*$
5	$9.00q^{*2}$	$3.0q^{*2}$	$4.00q^{*2}$	$2.78q^{*2}$	$3.0q^*$	q^*	$2.00q^*$	$1.67q^*$
20	$110.25q^{*2}$	$10.5q^{*2}$	$33.06q^{*2}$	$3.63q^{*2}$	$10.5q^*$	q^*	$5.75q^*$	$1.91q^*$

Table 2: Individual payoffs and outputs at different numbers of firms

5. General Analysis: Different Number of Privilege Stages

We extend the analysis on ICT in a more general format that allows each firm to have (i) different number of privilege stages and (ii) different discount factor. In this way, we can study in greater perspective the extent of possibilities to which an intertemporal collusion can be sustained in equilibrium. We begin by stating some of our assumptions:

1. Firms produce simultaneously a homogeneous good and obtain a payoff function defined in (11).
2. Set $q_i \in [0, q^{\max}]$ where $q^{\max} = \sup\{q_i \mid p(q_i, q_{-i}) > c\}$[6]
3. The number of firms in the oligopoly is fixed and at least 5.
4. The one-shot game Γ^1 has a symmetric pure strategy Cournot-Nash equilibrium
5. There is perfect monitoring in Γ^∞.

a. Uniform discount factor

Let k_i be the number of consecutive privilege stages that is given to the i^{th} player and denote r as the total number of k_is, for all i, in a single round , *i.e.* $k_1 + k_2 + ... + k_n = r$. In this general setup, each firm i regularly produces q^* but has the opportunity to produce $\bar{q}$ for k_i consecutive times after all the other firms before him

[6] Firms will only produce for as long as profit is positive, otherwise it will produce 0. This therefore does not entertain the possibility of enduring initial losses to obtain monopoly profits in the future.

have all made their turns. Once i's turn is finished, he goes back to producing q^* and waits for $r - k_i$ stages to be completed before he takes on again his privilege stages, and so on. For example, suppose $n = 3$ and $k_A = 1$, $k_B = 2$, and $k_C = 3$. Then, we see in Table 3 the intertemporal profile of production for the three players in the absence of any deviation:

	Stages									
Firms	1	2	3	4	5	6	7	8	9	...
A	$\bar{q}$	q^*	q^*	q^*	q^*	q^*	$\bar{q}$	q^*	q^*	...
B	q^*	$\bar{q}$	$\bar{q}$	q^*	q^*	q^*	q^*	$\bar{q}$	$\bar{q}$	...
C	q^*	q^*	q^*	$\bar{q}$	$\bar{q}$	$\bar{q}$	q^*	q^*	q^*	...

Table 3: An example of a production profile of a 3-player generalized ICT

This type of program for each firm is formally defined below where again any deviation is responded by a Cournot-Nash production forever after.

Definition 4. A generalized ICT strategy is a strategy profile $\{\mathbf{s}^t(h^t(\mathbf{q}^1, \mathbf{q}^2, ...\mathbf{q}^{t-1}))\}_{t=1}^{\infty}$ where:

(i) each firm $i \in N = \{1, 2, 3, ..., n\}$ precommits himself at the start of the game to a production profile $q_i(t)$ defined as follows:

$$q_i(t) = \begin{cases} \bar{q}\ , & \text{for all } t = \begin{cases} A + 1 + rz \\ A + 2 + rz \\ \vdots \\ A + k_i + rz \end{cases} \quad \begin{array}{c} \text{where } A = k_1 + k_2 + ... + k_{i-1}\ ; \\ z \in \{0, 1, 2, 3, ...\} \text{ and that} \\ \text{when } i = 1,\ \ A = 0. \end{array} \\ q^*, & \text{for all other time } t \text{ not defined for } \bar{q} \text{ production} \end{cases}$$

(ii) and if at all $t' < t$, where $t \geq 2$, $s_i^{t'}(h^{t'}) = q_i(t')$ for all $i \in N$, then $s_i^t(h^t) = q_i(t)$. Otherwise, $s_i^t(h^t) = q_c$ for all $i \in N$.

In the absence of any deviation from the pre-game commitment, the entire-game average discounted profit for each player i is given by

$$\Pi_i = \left(1-\delta^{\Sigma_{t=0}^{i-1}k_t}\right)\pi' + \left(\delta^{\Sigma_{t=0}^{i-1}k_t}(1-\delta^{k_i})\bar{\pi} + \delta^{\Sigma_{t=0}^{i}k_t}(1-\delta^{r-k_i})\pi'\right)\frac{1}{1-\delta^r}$$
$$= \pi' + \frac{(\bar{\pi}-\pi')(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t}}{(1-\delta^r)} \quad (12)$$

where $i \in N$ and $k_0 = 0$.

We now present two lemmas that are generalizations of Lemma 1 and 2.

Lemma 4.
(i) When n is sufficiently high, the IRC under a generalized ICT is always satisfied for all firms.
(ii) $\lim\limits_{\delta\to 1}\Pi_i = \frac{k_i}{r}\bar{\pi} + \frac{r-k_i}{r}\pi'$.

Proof:

(i) For the IRC to be satisfied, it must be that $\Pi_i > \Pi^c = \pi^c$ by definition. By using (12), we obtain

$$\frac{(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t}}{(1-\delta^r)} > \frac{\pi^c - \pi'}{\bar{\pi}-\pi'} \quad (13)$$

Since by assumption that $n \geq 5$, we know from Lemma 3 that $\pi' > \pi^c$. Thus, (13) is always satisfied for any firm i since its left-hand side can either be only positive or 0, for any $\delta \in (0,1)$ and $k_i > 0$.

(ii) Note that $\lim\limits_{\delta\to 1}\frac{1-\delta^a}{1-\delta^b} = \frac{a}{b}$. Applying this on (12) with $\delta \to 1$, we obtain the above result *q.e.d.*

Lemma 5. The ICC for a generalized ICT strategy with different number of privilege stages is the same for all i and is characterized by the following inequality:

$$\frac{(\pi^d-\pi')}{(\bar{\pi}-\pi')} - \delta\frac{(\pi^d-\pi^c)}{(\bar{\pi}-\pi')} < \delta\frac{(1-\delta^{k_i})}{(1-\delta^r)}.$$

For $\delta > \frac{\pi^d-\pi'}{\pi^d-\pi^c}$, the ICC is always satisfied.

Proof:

First, observe that firm i will not deviate during privilege stages where it receives $\bar{\pi}$. If i were to deviate, he will get a payoff less than $\bar{\pi}$ since $\bar{\pi}$ is already the maximum payoff i could get when all the others are producing q^*, *i.e.* $\bar{\pi} = \sup \ \pi_i\left(q_i, q_{-i}^*\right)$. Moreover, after the deviation, i would only receive π^c thereafter which is less than the interplay of earnings between π' and $\bar{\pi}$ under ICT (from Lemma 3).

Now, consider the regular stages. The condition not to deviate during these stages is depicted by $\Pi_i > \Pi^d(w)$, where Π_i is defined by (12) and $\Pi^d(w)$ is defined below:

$$\begin{aligned}\Pi^d(w) &= \pi'(1-\delta^{\Sigma_{t=0}^{i-1}k_t}) + \bar{\pi}\delta^{\Sigma_{t=0}^{i-1}k_t}(1-\delta^{k_i}) + \pi'\delta^{\Sigma_{t=0}^{i}k_t}(1-\delta^{w}) \\ &\quad + \pi^d\delta^{\Sigma_{t=0}^{i}k_t+w}(1-\delta) + \pi^c\delta^{\Sigma_{t=0}^{i}k_t+w+1} \end{aligned} \tag{14}$$

where $w \in \{0, 1, ..., (r - k_i - 1)\}$ and $r > k_i > 0$, for all i.

As argued in the proof of Lemma 2 (step 2), $\Pi^d(w) = \Pi^d(w+nz)$ which makes (14) to hold in all the stages of the infinitely-repeated game. The subsequent steps for the derivation of ICC are analogous to the proof presented in Step 3 of Lemma 2 and are presented below for the sake of completeness:

$$\begin{aligned}\pi' + \frac{(\bar{\pi}-\pi')(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t}}{1-\delta^r} &> \pi'(1-\delta^{\Sigma_{t=0}^{i-1}k_t}) + \bar{\pi}\delta^{\Sigma_{t=0}^{i-1}k_t}(1-\delta^{k_i}) + \pi'\delta^{\Sigma_{t=0}^{i}k_t}(1-\delta^{w}) \\ &\quad + \pi^d\delta^{\Sigma_{t=0}^{i}k_t+w}(1-\delta) + \pi^c\delta^{\Sigma_{t=0}^{i}k_t+w+1} \\ &= \pi' + (\bar{\pi}-\pi')\delta^{\Sigma_{t=0}^{i-1}k_t} - (\bar{\pi}-\pi')\delta^{\Sigma_{t=0}^{i}k_t} \\ &\quad + (\pi^d-\pi')\delta^{\Sigma_{t=0}^{i}k_t+w} - (\pi^d-\pi^c)\delta^{\Sigma_{t=0}^{i}k_t+w+1} \\ &= \pi' + (\bar{\pi}-\pi')(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t} \\ &\quad + \delta^{\Sigma_{t=0}^{i}k_t+w}[(\pi^d-\pi') - (\pi^d-\pi^c)\delta]\end{aligned}$$

By rearranging the terms, we obtain

$$\begin{aligned}(\bar{\pi}-\pi')(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t}\frac{\delta^r}{1-\delta^r} &> \delta^{\Sigma_{t=0}^{i}k_t+w}(\pi^d-\pi^c)\left(\frac{(\pi^d-\pi')}{(\pi^d-\pi^c)} - \delta\right) \\ \Leftrightarrow \delta^r\frac{(1-\delta^{k_i})}{(1-\delta^r)} &> \delta^{k_t+w}\left(\frac{(\pi^d-\pi')}{(\bar{\pi}-\pi')} - \frac{(\pi^d-\pi^c)}{(\bar{\pi}-\pi')}\delta\right)\end{aligned}$$

Notice that the last inequality is always satisfied whenever $\delta > \frac{\pi^d - \pi'}{\pi^d - \pi^c}$. Moreover, this condition is tight when $w = r - k_i - 1$, *i.e.* $\Pi^d(w)$ is highest. By substituting the value of w, we obtain the desired result. *q.e.d.*

The next lemma shall provide us an idea as to when a firm can obtain profits under (generalized) ICT that is higher than what the normal collusion can offer.

Lemma 6. Under the generalized ICT strategy, $\Pi_i > \Pi^*$ is equivalent to $\delta^{\Sigma_{t=0}^{i-1} k_t} \frac{1-\delta^{k_i}}{1-\delta^r} > \frac{2}{n+1}$, for all i, where $\delta \in (0,1)$ and $k_i > 0$. Moreover, as $\delta \to 1$, this condition becomes $\frac{k_i}{r} > \frac{2}{n+1}$.

Proof:

Recall that $\Pi^* = \pi^* = nq^{*2}$. From (12), we say that $\Pi_i > \Pi^*$ is equivalent to:

$$\pi' + \frac{(\bar{\pi} - \pi')(1 - \delta^{k_i})\delta^{\Sigma_{t=0}^{i-1} k_t}}{1 - \delta^r} > nq^{*2}$$

By substituting the values of $\bar{\pi}$ and π', we obtain:

$$q^{*2}\frac{(n+1)}{2} + q^{*2}\left(\frac{(n+1)^2}{4} - \frac{(n+1)}{2}\right)\frac{(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1} k_t}}{1-\delta^r} > nq^{*2}$$

$$\Leftrightarrow \quad \delta^{\Sigma_{t=0}^{i-1} k_t}\frac{1-\delta^{k_i}}{1-\delta^r} > \frac{2}{n+1} \tag{15}$$

By getting the limit of the left-hand side as $\delta \to 1$, we obtain the desired result. *q.e.d.*

Proposition 3. For a sufficiently high δ, a generalized ICT strategy

(i) is sustainable in a subgame-perfect equilibrium and

(ii) can allow some firms to obtain payoffs higher than the collusive profits Π^*.

Proof:

(i) The proof is immediate from the results of Lemma 4(i) and Lemma 5.

(ii) We want to show that there exists some i such that $\Pi_i > \Pi^*$. Let's take the case where $\delta \to 1$. From Lemma 6, the above-collusive profit condition is $\frac{k_i}{r} > \frac{2}{n+1}$.

To check that this condition is admissible for all i , it must be that $\frac{k_i}{r} + \sum\limits_{j\in N\backslash\{i\}} \frac{k_j}{r} = 1$. Then, we have $\frac{k_i}{r} = 1 - \sum\limits_{j\in N\backslash\{i\}} \frac{k_j}{r} > \frac{2}{n+1} \Rightarrow \sum\limits_{j\in N\backslash\{i\}} \frac{k_j}{r} < \frac{n-1}{n+1} \Rightarrow \frac{k_j}{r} < \frac{1}{n+1}$. Thus, for any $j \in N\backslash\{i\}$, it is still possible to obtain a $\frac{k_j}{r} > 0$. From Lemma 4(ii), we see that $\Pi_j \in (\pi', \bar{\pi})$ is greater than π^c for any $\frac{k_j}{r} > 0$, thus, IRC is satisfied for all i. Finally, we see from Lemma 5 that as $\delta \to 1$, the ICC condition is reduced to $\frac{\pi^c - \pi'}{\bar{\pi} - \pi'} < \frac{k_j}{r}$ which is always satisfied since $\pi^c < \pi'$ by the assumption of high n. This now completes our proof. *q.e.d.*

Note however that since $\sum_{i=1}^{n} \frac{k_i}{r} = 1$, not all firms can obtain a k_i/r ratio higher than $\frac{2}{n+1}$ when $\delta \to 1$ since $\frac{2}{n+1}n$ exceeds 1, for all $n > 1$. We generalize this assertion for any $\delta \in (0, 1)$ through the following claim.

Claim. In an oligopoly with n firms and with a uniform $\delta \in (0, 1)$, it is impossible to have $\Pi_i > \Pi^*$ for all i.

Proof:

Suppose it is possible. Then, the aggregate profit across firms is greater than the total payoff obtained in the normal collusion, *i.e.* $\sum_{i=1}^{n} \Pi_i > n\Pi^*$. Using (12), this implies that

$$n\pi' + \sum_{i=1}^{n} \frac{(\bar{\pi} - \pi')(1 - \delta^{k_i})\delta^{\Sigma_{t=0}^{i-1} k_t}}{(1 - \delta^r)} > n\Pi^* = n\pi^*$$

$$\Leftrightarrow \quad n\pi' + \frac{(\bar{\pi} - \pi')}{(1 - \delta^r)}[\left(1 - \delta^{k_1}\right) + \left(\delta^{k_1} - \delta^{k_1+k_2}\right) + \left(\delta^{k_1+k_2} - \delta^{k_1+k_2+k_3}\right) + \\ ... + \left(\delta^{k_1+k_2+...+k_{n-1}} - \delta^r\right)] > n\pi^*$$

$$\Leftrightarrow \quad \frac{(\bar{\pi} - \pi')}{(1 - \delta^r)}(1 - \delta^r) > n(\pi^* - \pi')$$

$$\Leftrightarrow \quad \frac{n^2 - 1}{4} > n\left(\frac{n-1}{2}\right), \quad \text{by substituting the values of } \bar{\pi}, \pi^*, \text{ and } \pi'$$

$$\Leftrightarrow \quad n < 1 \text{ , which is a contradiction}$$

q.e.d.

The above-collusive profits obtained by some are therefore made at the expense of some receiving below-collusive payoffs. In what follows, we show that it is nonetheless still possible to earn beyond the collusive profits for all firms, provided that the discount factor is differentiated among them.

b. Differentiated discount factor

The main result of this section is presented in the following proposition. Its proof is instructive as it gives a simple recursive algorithm on generating a sustainable generalized ICT that offers above-collusive income to all firms.

Proposition 4. Given $k_i > 0$ and $\delta_i \in (0, 1)$ for each firm i, it is possible to construct a subgame-perfect generalized ICT strategy where $\Pi_i > \Pi^*$, for all i.

Proof:

Start by setting $\delta_i \in \left(\frac{\pi^d - \pi'}{\pi^d - \pi^c}, 1\right)$ for all $i \in N$, where $N \geq 5$, in order that subgame-perfect equilibrium can be admitted. When firms have different discount factors, the result of Lemma 6 can be rewritten as $\delta_i^{\Sigma_{t=0}^{i-1} k_t} \frac{1-\delta_i^{k_i}}{1-\delta_i^r} > \frac{2}{n+1}$. This is similarly expressed in (16), where we let $r = \alpha n$, for some $\alpha \in \mathbb{Z}^+$. Note here that log δ_i is negative.

$$k_i > \frac{\log\left(1 - \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}}\right)}{\log\ \delta_i} \tag{16}$$

In order that $k_i > 0$, for all i, it must be that

$$0 < 1 - \frac{2\ (1 - \delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} < 1$$

Lemma 7, which is presented at the end of this proof, asserts that this is possible provided that α is set sufficiently high and that $\delta_i^{\Sigma_{t=0}^{i-1} k_t} > \frac{2}{n+1}$, for all i.

Given these conditions, we now define

$$\underline{k}_i = \inf\left\{ k_i \in \mathbb{Z}^+ \;\middle|\; k_i > \frac{\log\left(1 - \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} \underline{k}_t}}\right)}{\log\ \delta_i} \right\}.$$

Thus, $\underline{k}_i$ (an integer above zero) is the least number of privilege stages that can generate above-collusive income for i given its δ_i. We obtain the set of all $\underline{k}_i$s by starting with $\underline{k}_1$, given δ_1 and $\underline{k}_0 = 0$. Then, compute $\underline{k}_2$ using δ_2 and the derived $\underline{k}_1$, and so on. By choosing $\underline{k}_i$ as the k_i of each $i = \{1, 2, ..., n-1\}$ and $k_n = \alpha n - \sum_{i=1}^{n-1} \underline{k}_i$ whenever $k_n \geq \underline{k}_n$, we have constructed a set of k_is for a generalized ICT that is sustainable in equilibrium and that yields above-collusive payoffs for all i. However, if $k_n < \underline{k}_n$, then increase α and repeat the entire process of solving for $\underline{k}_i$s until $k_n \geq \underline{k}_n$ is satisfied *q.e.d.*

Lemma 7. If $\delta_i^{\Sigma_{t=0}^{i-1} k_t} > \frac{2}{n+1}$ for all i, where $k_1, k_2, ..., k_{i-1} > 0$ and $k_0 = 0$, and that α is set sufficiently high, then $1 - \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} \in (0, 1)$ for all i .

Proof:

First, we show that $1 - \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} > 0$. From the given, we have

$$\begin{aligned} 1 \quad &> \quad \frac{2}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} \quad \text{for any } i \in N \text{ and } \delta_i \in (0,1) \\ &> \quad \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} \end{aligned}$$

This is true because as the function $(1 - \delta_i^{\alpha n})$ monotonically decreases from 1 as α decreases, there exists $\bar{\alpha} \geq 1$ such that for any $\alpha \in (\bar{\alpha}, \infty)$ the above inequality continues to hold. Second, observe that the condition $1 - \frac{2\,(1-\delta_i^{\alpha n})}{(n+1)\,\delta_i^{\Sigma_{t=0}^{i-1} k_t}} < 1$ is always true for any $\delta_i \in (0, 1)$ and any finite n. *q.e.d.*

An Example:

Let $n = 5$ and set $\frac{k_1}{r} = \frac{4}{100}$, $\frac{k_2}{r} = \frac{7}{100}$, $\frac{k_3}{r} = \frac{18}{100}$, $\frac{k_4}{r} = \frac{34}{100}$, and $\frac{k_5}{r} = \frac{37}{100}$. Suppose also that $\delta_1 = 0.880$, $\delta_2 = 0.900$, $\delta_3 = 0.950$, $\delta_4 = 0.990$, and $\delta_5 = 0.999$. From Lemma 6, the condition $\Pi_i \geq \Pi^*$ is equivalent to $\delta_i^{\Sigma_{t=0}^{i-1} k_i} \frac{1-\delta_i^{k_i}}{1-\delta_i^{r}} > \frac{1}{3}$, for all i. Now, since $\delta_1^0 \frac{1-\delta_1^4}{1-\delta_1^{100}} = 0.40$, $\delta_2^4 \frac{1-\delta_2^7}{1-\delta_2^{100}} = 0.34$, $\delta_3^{11} \frac{1-\delta_3^{18}}{1-\delta_3^{100}} = 0.34$, $\delta_4^{29} \frac{1-\delta_4^{34}}{1-\delta_4^{100}} = 0.34$, and $\delta_5^{63} \frac{1-\delta_5^{37}}{1-\delta_5^{100}} = 0.36$, we see that the above condition is satisfied for all $i = \{1, 2, 3, 4, 5\}$. Finally, this set-up is subgame-perfect since by construction $\delta_i > \frac{\pi^d - \pi'}{\pi^d - \pi^c} = \frac{9}{11}$ for all i and for $n = 5$.

c. A case of duopoly

In principle, the ICT strategy in a duopoly should be easier to sustain given its lesser number of players. Both players do not have to wait so long for their privilege stages to come and so they do not need to have very high discount factors. However, the payoff structure under a duopoly is quite different from the general analysis we have discussed since its Cournot-Nash punishment does not pose as much deterrence to any potential deviant as in the case of a high-n oligopoly. Notice from Table 2 that when $n = 2$, the profit π^c under punishment phase is higher than both π^d and π' which could even provoke deviation from any firm, if not of the high $\bar{\pi}$ that serves as the only incentive to stick to the program. Consequently, one needs to have a more stringent IRC to neutralize the temptation posed by the high π^c. Moreover, the ICC requirement presented in Lemma 5 becomes also different in the sense that the left-hand side of inequality becomes always positive, thereby reducing the possible set for sustainable cooperation. Despite these differences, we show nonetheless that the two-firm case can still admit the results of Proposition 3.

For $n = 2$, the ICC in Lemma 5 is reduced to the following expression after substituting the values for $\bar{\pi}$, π^d, π', and π^c:

$$\frac{1}{12} - \delta_i \left(\frac{-31}{108} \right) < \delta_i^{r-k_i} \frac{(1 - \delta_i^{k_i})}{(1 - \delta_i^r)} \tag{17}$$

By setting $k_1 = k_2 = 1$ and $\delta_1 = \delta_2 = \delta$, we obtain $\delta \in \left(\frac{34-\sqrt{877}}{31}, 1 \right)$. To pass the IRC, player 2 has a more binding constraint and so Lemma 4 is reduced to

$$\frac{\delta}{1+\delta} > \frac{10}{27} \ ,$$

where it must be that $\delta \in \left(\frac{10}{17}, 1 \right)$. Since IRC has a more stringent constraint than ICC, the former is the binding constraint for δ that admits subgame perfection.

Unfortunately, this duopoly can not provide above-collusive income for firms even when set under different discount factors (*i.e.* Proposition 4). For example, for player 2 to have above-collusive income it must be that $\frac{(1-\delta_2)}{(1-\delta_2^2)}\delta_2 > \frac{2}{3}$, implying that $\delta_2 > 2$

which is impossible. In general, we show that this is not possible in any duopoly, as presented in the following proposition.

Proposition 5. If a generalized ICT strategy is sustainable in a duopoly with k_1, $k_2 > 0$ and $\delta_i \in (0, 1)$, then it is not possible that $\Pi_i > \Pi^*$ for both i.

Proof:

Suppose both players obtain above-collusive profits, then player 2 satisfies the condition stated in Lemma 6, *i.e.* $\delta_2^{k_1} \frac{(1-\delta_2^{k_2})}{(1-\delta_2^r)} > \frac{2}{3}$, where $r = k_1 + k_2$. Since the left-hand side is monotonically increasing in δ_2 and has a limit of $\frac{k_2}{r}$ as $\delta_2 \to 1$, then it must be that $\frac{k_2}{r} > \frac{2}{3}$. Thus, this implies that $\frac{k_1}{r} < \frac{1}{3}$ for player 1. Now examine the ICC condition for player 1. Since the right-hand side of (17) is also monotonically increasing in δ_1 and approaches $\frac{k_1}{r}$ as $\delta_1 \to 1$, then (17) is reduced to $\frac{k_1}{r} > \frac{10}{27}$, which is a contradiction. *q.e.d.*

6. Conclusion

We have shown in this paper that the ICT strategy, which is sustainable in perfect equilibrium, generates higher consumer and producer welfare than the normal intertemporal collusion. While it is straightforward to show how all consumers are made better off through lower price offer, it is not so for the producers since one cannot make some firms obtain higher than the normal collusive payoff without making others earn below it. We proved however that when firms have differentiated discount factors, then it is possible to form a mechanism where everyone receives payoff higher than the collusive outcome.

In summary, the ICT draws out greater possibilities for cooperation through an unconventional design of contract that seeks to award each firm at different stages of the game. The privileged incentive that each one obtains in the future stages is sufficient for everyone not to abandon the commitment, even if the penalty is not made more severe. This we showed by maintaining only a Counot-Nash punishment while exploring the various forms of stable contracts characterized by the generalized ICT strategy.

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Appendix

A.

In extending further the study on ICT, we present here Abreu’s carrot-and-stick punishment scheme (1986) to replace the less severe yet irreversible Cournot-Nash penalty which we have used throughout the text. The main significance of this penal structure, eventhough it may be more complex, is that it serves as an antidote to any possible *ex post* renegotiation that tries to recover collusive potentials lost after a deviation.

We start by defining Abreu's one-time intense punishment which when performed brings back the game in the next period to its original ICT path. Let

$$\hat{A} = (1-\delta)\pi^0 + \delta\Pi_i,$$

where π^0 (can be set to 0) is the one shot punishment payoff associated from producing q^0. Denote also $\tilde{\pi}^0$ as the maximum payoff obtained from deviating unilaterally from this punishment path such that

$$\tilde{\pi}^0 = \arg\max_{q_i \geq 0} \pi_i \left(q_i, q_{-i}^0\right).$$

We define Abreu's carrot-and-stick strategies below, noting that the only way to go back to the ICT path after a deviation has been done is when everyone executes the costly single period punishment production q^0.

(i) Every firm i precommits at the start of the game to a production profile:

$$q_i(t) = \begin{cases} \bar{q}\ , & \text{for all } t = i + nz, \text{ where } z \in \{0,1,2,...\} \\ q^*, & \text{for all } t \neq i + nz, \text{ where } z \in \{0,1,2,...\} \end{cases}$$

(ii) and if at all $t' < t$, where $t \geq 2$, $s_i^{t'}(h^{t'}) = q_i(t')$ for all $i \in N$, then $s_i^t(h^t) = q_i(t)$

(iii) also, if at all $t' < t$, where $t \geq 2$, $s_i^{t'}(h^{t'}) = q_i^0(t')$ for all $i \in N$, then $s_i^t(h^t) = q_i(t)$.

(iv) Otherwise, $s_i^t(h^t) = q_i^0$ for all $i \in N$.

In order for the punishment scheme to be in equilibrium, no firm should be willing to deviate once the game enters into this phase, i.e.

$$\hat{A} \geq (1-\delta)\tilde{\pi}^0 + \delta\hat{A} \Leftrightarrow \hat{A} \geq \tilde{\pi}^0.$$

Then, we ensure also that firms do not deviate from the precommitted path given this penalty scheme. We show this needed condition as follows:

$$\begin{aligned}\Pi_i &> \pi'(1-\delta^{\Sigma_{t=0}^{i-1}k_t}) + \bar{\pi}\delta^{\Sigma_{t=0}^{i-1}k_t}(1-\delta^{k_i}) + \pi'\delta^{\Sigma_{t=0}^{i}k_t}(1-\delta^{w}) \\ &\quad +\pi^d\delta^{\Sigma_{t=0}^{i}k_t+w}(1-\delta) + \pi^0\delta^{\Sigma_{t=0}^{i}k_t+w+1}(1-\delta) + \Pi_i\delta^{\Sigma_{t=0}^{i}k_t+w+2} \\ &= \pi' + (\bar{\pi}-\pi')\delta^{\Sigma_{t=0}^{i-1}k_t} - (\bar{\pi}-\pi')\delta^{\Sigma_{t=0}^{i}k_t} + (\pi^d-\pi')\delta^{\Sigma_{t=0}^{i}k_t+w} \\ &\quad -(\pi^d-\pi^0)\delta^{\Sigma_{t=0}^{i}k_t+w+1} - \pi^0\delta^{\Sigma_{t=0}^{i}k_t+w+2} + \Pi_i\delta^{\Sigma_{t=0}^{i}k_t+w+2}\end{aligned}$$

By rearranging the terms, and given that $\pi' + (\bar{\pi}-\pi')(1-\delta^{k_i})\delta^{\Sigma_{t=0}^{i-1}k_t} = \Pi_i(1-\delta^r) + \pi'\delta^r$ and that $w = r - k_i - 1$, we have

$$\begin{aligned}\Pi_i(\delta^r - \delta^{\Sigma_{t=0}^{i}k_t+w+2}) - \pi'\delta^r &> \delta^{\Sigma_{t=0}^{i}k_t+w}[(\pi^d-\pi') - (\pi^d-\pi^0)\delta - \pi^0\delta^2] \\ \Rightarrow \Pi_i(1-\delta^{\Sigma_{t=0}^{i-1}k_t+1}) - \pi' &> \delta^{\Sigma_{t=0}^{i-1}k_t-1}[(\pi^d-\pi') - (\pi^d-\pi^0)\delta - \pi^0\delta^2]\end{aligned}$$

Then, by substituting the value of Π_i, we obtain

$$\begin{aligned}\frac{(\bar{\pi}-\pi')(1-\delta^{k_i})(1-\delta^{\Sigma_{t=0}^{i-1}k_t+1})}{1-\delta^r} &> \pi'\frac{(\delta^2-1)}{\delta} + \pi^d\frac{(1-\delta)}{\delta} + \pi^0(1-\delta) \\ \Rightarrow \frac{(1-\delta^{k_i})(1-\delta^{\Sigma_{t=0}^{i-1}k_t+1})}{(1-\delta^r)(1-\delta)}\delta &> \frac{\pi^d-\pi'}{\bar{\pi}-\pi'} - \frac{\pi'-\pi^0}{\bar{\pi}-\pi'}\delta\end{aligned}$$

Finally, note that as $\delta \to 1$, the last inequality turns into the following simplified condition:

$$\frac{k_i}{r}(k_1 + k_2 + ... + k_{i-1} + 1) > \frac{\pi^d - 2\pi' + \pi^0}{\bar{\pi}-\pi'}$$

B.

Lemma 8. Denote as $\left\{\left\langle\rho_{(s)}\right\rangle_{s=1}^{l}\right\}_{c=1}^{\infty}$ the ordered sequence of payoffs $\left\langle\rho_{(s)}\right\rangle$ of length l which is cyclically repeated infinitely. The condition not to deviate from this path at the m^{th} stage of the sequence is equivalent to any of the $(m+(r-1)l)^{th}$ stage, where $r \in \mathbb{Z}^+$ is the cycle number.

Proof.

Denote ξ_∞ as the sum of the payoffs of the profile $\left\{\left\langle\rho_{(s)}\right\rangle_{s=1}^{l}\right\}_{c=1}^{\infty}$. For a deviation at the m^{th} stage of the first cycle, we depict its entire-game profile as $\left\langle\left\langle\rho_{(s)}\right\rangle_{s=1}^{m-1}, \rho_m, \{\hat{\rho}\}_{s=m+1}^{\infty}\right\rangle$ and denote its sum as $\xi_{\langle m,1\rangle}$. Note from this profile that ρ_m is the deviatory payoff at stage m and $\hat{\rho}$ is the subsequent punishment payoff obtained at every stage. Now, if a deviation occurs in any of the succeeding cycle $r \in \mathbb{Z}^+\backslash\{1\}$, the profile is depicted as follows with the sum denoted as $\xi_{\langle m,r\rangle}$:

$$\left\langle \left\{ \left\langle \rho_{(s)} \right\rangle_{s=1}^{l} \right\}_{c=1}^{r-1}, \left\langle \rho_{(s)} \right\rangle_{s=(r-1)l+1}^{m-1+(r-1)l}, \rho_{m+(r-1)l}, \{\hat{\rho}\}_{s=m+1+(r-1)l}^{\infty} \right\rangle .$$

To prove this lemma, we show that the difference between ξ_∞ and $\xi_{\langle m,1\rangle}$ is equivalent to ξ_∞ and $\xi_{\langle m,r\rangle}$ for any $r \in \mathbb{Z}^+\backslash\{1\}$, i.e. $d\left(\xi_\infty, \xi_{\langle m,1\rangle}\right) = d\left(\xi_\infty, \xi_{\langle m,r\rangle}\right)$. First, define $\Xi : \langle\cdot\rangle \to \mathbb{R}$ as a mapping of the payoff profile $\langle\cdot\rangle$ to its aggregate sum. Then, we have

$$\begin{aligned}
\xi_{\langle m,r\rangle} &\equiv \Xi \left\langle \left\{ \left\langle \rho_{(s)} \right\rangle_{s=1}^{l} \right\}_{c=1}^{r-1}, \left\langle \rho_{(s)} \right\rangle_{s=(r-1)l+1}^{m-1+(r-1)l}, \rho_{m+(r-1)l}, \{\hat{\rho}\}_{s=m+1+(r-1)l}^{\infty} \right\rangle \\
&\equiv \Xi \left\langle \left\{ \left\langle \rho_{(s)} \right\rangle_{s=1}^{l} \right\}_{c=1}^{r-1}, \delta^{(r-1)l} \xi_{\langle m,1\rangle} \right\rangle \\
&\equiv \delta^{(r-1)} \Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_{\langle m,1\rangle} \right\rangle
\end{aligned}$$

Similarly, we can express ξ_∞ as follows

$$\begin{aligned}
\xi_\infty &\equiv \Xi \left\langle \left\{ \left\langle \rho_{(s)} \right\rangle_{s=1}^{l} \right\}_{c=1}^{\infty} \right\rangle \\
&\equiv \Xi \left\langle \left\{ \left\langle \rho_{(s)} \right\rangle_{s=1}^{l} \right\}_{c=1}^{r-1}, \delta^{(r-1)l} \xi_\infty \right\rangle \\
&\equiv \delta^{(r-1)} \Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_\infty \right\rangle
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
d\left(\xi_\infty, \xi_{\langle m,r\rangle}\right) &= d\left(\delta^{(r-1)} \Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_\infty \right\rangle,\ \delta^{(r-1)} \Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_{\langle m,1\rangle} \right\rangle\right) \\
&= d\left(\Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_\infty \right\rangle,\ \Xi \left\langle \left\langle \rho_{(s)} \right\rangle_{s=1}^{l}, \delta^{l} \xi_{\langle m,1\rangle} \right\rangle\right) \\
&= d\left(\xi_\infty,\ \xi_{\langle m,1\rangle}\right)
\end{aligned}$$

q.e.d.